Fuzzy Rough Sets: Beyond the Obvious

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***Abstract*— Rough set theory was introduced in 1982. Soon it was combined with fuzzy set theory, giving rise to a hybrid model, involving fuzzy sets and fuzzy relations, which appears to be a natural, elegant generalization. In this paper we reveal that in the fuzzification process an important step seems to be overlooked. The most fascinating part is that this forgotten step arises from the true essence of fuzzy set theory: namely, that an element can belong to a given degree to more than one fuzzy set at the same time.**

I. Introduction

Pawlak [12] launched rough set theory as a framework for the construction of approximations of concepts when only incomplete information is available. The available information consists of a set A of examples (a subset of a universe X, X being a non-empty set of objects we want to say something about) of a concept C, and a relation R in X. R models "indiscernibility" or "indistinguishability" and therefore generally is a tolerance relation (i.e. a reflexive and symmetrical relation) and in most cases even an equivalence relation (i.e. a transitive tolerance relation). Rough set analysis makes statements about the membership of some element y of X to the concept C of which A is a set of examples, based on the indistinguishability between y and the elements of A. To arrive at such statements, A is approximated in two ways. An element y of X belongs to the lower approximation of A if the equivalence class to which y belongs is included in A. On the other hand y belongs to the upper approximation of A if its equivalence class has a non–empty intersection with A.

After a public debate reflecting rivalry between this new theory and the slightly older fuzzy set theory, many people have worked on the fuzzification of upper– and lower approximations (e.g. [6], [11], [14], [15], [17]). In doing so, the central focus moved from elements' indistinguishability (w.r.t. their attribute values in an information system) to their similarity—represented by a fuzzy relation R—again w.r.t. to those attribute values: objects are categorized into classes with "soft" boundaries based on their similarity to one another. A concrete advantage of such a scheme is that abrupt transitions between classes are replaced by gradual ones, allowing that an element can belong (to varying degrees) to more than one class. On another count the set A to be approximated can be fuzzy as well in the new hybrid model, which is called "fuzzy rough set theory".

The most striking aspect of all the studies mentioned above is that none of them tries to exploit the fact that an element y of X can belong to some degree to several "soft similarity classes" at the same time. This property does not only lie at the heart of fuzzy set theory but is also crucial in the decision on how to define lower and upper approximations. For instance, as mentioned above, in traditional rough set theory, y belongs to the lower approximation of A if the equivalence class to which y belongs is included in A. But what happens if y belongs to several "fuzzy equivalence classes" at the same time? Do we then require that all of them are included in A? Most of them? Or just one? And then, which one?

Traditional fuzzy rough set theory involves only one fuzzy equivalence class. In this paper we explore what happens if we abandon this most obvious choice. After recalling the necessary preliminaries in Section 2, in Section 3 we define alternative lower and upper approximations of a (fuzzy) set A, based on different choices about which fuzzy equivalence classes should be included in, or have a non-empty intersection, with A. In Section 4 we examine their properties, paying significant attention to the role that the $\mathcal{T}$-transitivity of the fuzzy relation R plays in this game. This allows us to end with an interesting conclusion and some ideas for further research.

II. Preliminaries

Throughout this paper, let $\mathcal{T}$ and $\mathcal{I}$ denote a triangular norm and an implicator respectively. Recall that a triangular norm (t–norm for short) $\mathcal{T}$ is any increasing, commutative and associative $[0,1]^2 \rightarrow [0,1]$ mapping satisfying $\mathcal{T}(1,x) = x$, for all x in $[0,1]$. A negator $\mathcal{N}$ is a decreasing $[0,1] \rightarrow [0,1]$ mapping satisfying $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$. $\mathcal{N}$ is called involutive if $\mathcal{N}(\mathcal{N}(x)) = x$ for all x in $[0,1]$. Finally, an implicator is any $[0,1]^2 \rightarrow [0,1]$–mapping $\mathcal{I}$ satisfying $\mathcal{I}(0,0) = 1, \mathcal{I}(1,x) = x$, for all x in $[0,1]$. Moreover we require $\mathcal{I}$ to be decreasing in its first, and increasing in its second component. If $\mathcal{T}$ is a t–norm, the mapping $\mathcal{I}_\mathcal{T}$ defined by, for all x and y in [0,1],

$$\mathcal{I}_\mathcal{T}(x,y) = \sup\{\lambda | \lambda \in [0,1] \text{ and } \mathcal{T}(x,\lambda) \leq y\}$$

is an implicator, usually called the residual implicator (of $\mathcal{T}$). If $\mathcal{T}$ is a t–norm and $\mathcal{N}$ is an involutive negator, then the mapping $\mathcal{I}_{\mathcal{T},\mathcal{N}}$ defined by, for all x and y in [0,1],

$$\mathcal{I}_{\mathcal{T},\mathcal{N}}(x,y) = \mathcal{N}(\mathcal{T}(x, \mathcal{N}(y)))$$

is an implicator, usually called the S–implicator induced by $\mathcal{T}$ and $\mathcal{N}$. In Table I, II and III, we mention some well–known t–norms, S– and residual implicators. The S–implicators in Table II are induced by means of the standard negator $\mathcal{N}_s$ which is defined by $\mathcal{N}_s(x) = 1 - x$, for all x in $[0, 1]$.

TABLE I

WELL-KNOWN T–NORMS (x AND y IN $[0, 1]$)

$\mathcal{T}_{\mathrm{M}}(x, y)$	$=$	$\min(x, y)$
$\mathcal{T}_{\mathrm{P}}(x, y)$	$=$	xy
$\mathcal{T}_{\mathrm{W}}(x, y)$	$=$	$\max(x + y - 1, 0)$

TABLE II

WELL-KNOWN S–IMPLICATORS (x AND y IN $[0, 1]$)

$\mathcal{I}_{\mathcal{T}_{\mathrm{M}},\mathcal{N}_s}(x, y)$	$=$	$\max(1 - x, y)$
$\mathcal{I}_{\mathcal{T}_{\mathrm{P}},\mathcal{N}_s}(x, y)$	$=$	$1 - x + xy$
$\mathcal{I}_{\mathcal{T}_{\mathrm{W}},\mathcal{N}_s}(x, y)$	$=$	$\min(1 - x + y, 1)$

TABLE III

WELL-KNOWN RESIDUAL IMPLICATORS (x AND y IN $[0, 1]$)

$\mathcal{I}_{\mathcal{T}_{\mathrm{M}}}(x, y)$	$=$	$\begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}$
$\mathcal{I}_{\mathcal{T}_{\mathrm{P}}}(x, y)$	$=$	$\begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{otherwise} \end{cases}$
$\mathcal{I}_{\mathcal{T}_{\mathrm{W}}}(x, y)$	$=$	$\min(1 - x + y, 1)$

Both a t–norm and its residual implicator as well as a t–norm and an S–implicator induced by it, satisfy the weak shunting principle [14]:

$$\mathcal{I}(\mathcal{T}(x, y), z) \leq \mathcal{I}(x, \mathcal{I}(y, z)) \quad (1)$$

Many other pairs of t–norms and implicators satisfy this principle as well. In fact, if $\mathcal{T}$ and $\mathcal{I}$ satisfy it, then it also holds for $\mathcal{I}$ combined with every larger t-norm $\mathcal{T}_1$ (i.e. $\mathcal{T}_1(x, y) \geq \mathcal{T}(x, y)$, for all x and y in $[0, 1]$), because of the decreasingness of $\mathcal{I}$ in its first component. This means that $\mathcal{T}_{\mathrm{M}}$, which is the largest t-norm, satisfies the weak shunting principle with every residual or S–implicator. Hence it is very easy to find pairs of t–norms and implicators satisfying this principle, although it does not hold in general for arbitrary combinations as the following example shows.

Example 1: For $x = 0.5, y = 0.7$ and $z = 0.1$

$$\begin{aligned} \mathcal{I}_{\mathcal{T}_{\mathrm{P}}}(\mathcal{T}_{\mathrm{W}}(x, y), z) &= \mathcal{I}_{\mathcal{T}_{\mathrm{P}}}(0.2, 0.1) = 0.5 \\ \mathcal{I}_{\mathcal{T}_{\mathrm{P}}}(x, \mathcal{I}_{\mathcal{T}_{\mathrm{P}}}(y, z)) &= \mathcal{I}_{\mathcal{T}_{\mathrm{P}}}(0.5, \frac{1}{7}) = \frac{2}{7} \end{aligned}$$

Hence $\mathcal{I}_{\mathcal{T}_{\mathrm{P}}}(\mathcal{T}_{\mathrm{W}}(x, y), z) > \mathcal{I}_{\mathcal{T}_{\mathrm{P}}}(x, \mathcal{I}_{\mathcal{T}_{\mathrm{P}}}(y, z))$.

Recall that a fuzzy set A in a universe X is an $X \to [0, 1]$ mapping. Its height is defined as $\mathrm{hgt} A = \sup\{A(x) | x \in X\}$. The $\mathcal{T}$–intersection of fuzzy sets A and B in X is the fuzzy set $A \cap_{\mathcal{T}} B$ defined by

$$A \cap_{\mathcal{T}} B(x) = \mathcal{T}(A(x), B(x))$$

for all x in X. The degree of overlap of A and B and the degree of inclusion of A in B are defined by

$$\begin{aligned} \textsc{Overl}(A, B) &= \sup_{x \in X} \mathcal{T}(A(x), B(x)) \\ \textsc{Incl}(A, B) &= \inf_{x \in X} \mathcal{I}(A(x), B(x)) \end{aligned}$$

A binary fuzzy relation R in X is a fuzzy set in $X \times X$. For all y in X, the R-foreset of y is the fuzzy set Ry defined by $Ry(x) = R(x, y)$ for all x in X. If for all x in X, $A(x) \in \{0, 1\}$ then A is called a (crisp) set. Likewise if for all x and y in X, $R(x, y) \in \{0, 1\}$ then R is called a (crisp) relation.

III. BEYOND THE OBVIOUS

A. Fuzzy Relations

Fuzzy $\mathcal{T}$-equivalence relations are the commonly used generalization of crisp equivalence relations.

Definition 2: A binary fuzzy relation R in X is called a fuzzy $\mathcal{T}$-equivalence relation iff for all x, y and z in X

(FE.1) $R(x, x) = 1$ (reflexivity)
(FE.2) $R(x, y) = R(y, x)$ (symmetry)
(FE.3) $\mathcal{T}(R(x, y), R(y, z)) \leq R(x, z)$ ($\mathcal{T}$–transitivity)

Because crisp equivalence relations are used to model equality, fuzzy $\mathcal{T}$-equivalence relations are commonly considered to represent approximate equality or similarity. We will comment on this later. Let us first observe that the R–foresets of a crisp equivalence relation coincide with its equivalence classes, i.e. for all y in X, Ry is the equivalence class to which y belongs, often denoted by $[y]_R$. It is well known that in the crisp case, if we consider two equivalence classes then they either coincide or are disjoint. It is therefore not possible for y to belong to two different equivalence classes at the same time. If R is a fuzzy relation on X—in particular a $\mathcal{T}$–fuzzy equivalence relation —then it is quite normal that, because of the intermediate degrees of membership, different foresets are not necessarily disjoint.

Example 3: One can verify that for the fuzzy $\mathcal{T}$–equivalence relation R on $X = \{a, b\}$

R	a	b
a	1	0.2
b	0.2	1

it holds that

$$\begin{aligned} Ra &= \{(a, 1), (b, 0.2)\} \\ Rb &= \{(a, 0.2), (b, 1)\} \end{aligned}$$

hence, for any t–norm $\mathcal{T}$,

$$Ra \cap_{\mathcal{T}} Rb = \{(a, 0.2), (b, 0.2)\}$$

Note that, although $Rx \cap_{\mathcal{T}} Ry = \emptyset$ clearly does not hold in general if $x \neq y$, we do have the following property which is a generalization of [7].

Proposition 4: If R is a fuzzy $\mathcal{T}$–equivalence relation in X then for all x and y in X

$$\mathrm{hgt}(Rx \cap_{\mathcal{T}} Ry) \leq R(x, y)$$

Hence,

$$R(x, y) = 0 \text{ implies } Rx \cap_{\mathcal{T}} Ry = \emptyset$$

In other words the disjointness of R–foresets is preserved if the elements are entirely unrelated (i.e. related to degree 0). On the other hand the coincidence of R-foresets is preserved provided that the elements are fully related to each other (i.e. related to degree 1).

Proposition 5: [6] If R is a fuzzy $\mathcal{T}$–equivalence relation in X then for all x and y in X

$$R(x, y) = 1 \Rightarrow Rx = Ry$$

As a consequence fuzzy $\mathcal{T}$–equivalence relations are not compatible with the Poincaré paradox [5]. We say that a fuzzy relation E in a universe X, containing at least three elements, is compatible with the Poincaré paradox if

$$(\exists(x, y, z) \in X^3)(E(x, y) = 1 \wedge E(y, z) = 1 \wedge E(x, z) < 1)$$

This is inspired by Poincaré's [9] experimental observation that a bag of sugar of 10 grammes and a bag of 11 grammes can be perceived as indistinguishable by a human being. The same applies for a bag of 11 grammes w.r.t. a bag of 12 grammes, while the subject is perfectly capable of noting a difference between the bags of 10 and 12 grammes. Now if E is a fuzzy $\mathcal{T}$–equivalence relation, then according to Proposition 5, $E(x, y) = 1$ implies $Ex = Ey$. Since $Ey(z) = E(y, z) = 1$, also $Ex(z) = E(x, z) = 1$ which is in conflict with $E(x, z) < 1$. The fact that they are not compatible with the Poincaré paradox makes fuzzy $\mathcal{T}$–equivalence relations less suited to model approximate equality. The main underlying cause for this conflict is $\mathcal{T}$–transitivity. This is why we will make it very explicit in this paper every time we rely on $\mathcal{T}$–transitivity of the fuzzy relation involved.

B. Alternative Approximations

Let X be a universe and R a crisp equivalence relation. The lower and the upper approximation (in the sense of Pawlak [12]) of a crisp subset A of X in the approximation space (X, R) are the crisp sets $R{\downarrow}A$ and $R{\uparrow}A$ such that for all y in X

$$\begin{aligned} y \in R{\downarrow}A \quad &\text{iff} \quad Ry \subseteq A \\ y \in R{\uparrow}A \quad &\text{iff} \quad Ry \cap A \neq \emptyset \end{aligned}$$

In other words

$$y \in R{\downarrow}A \quad \text{iff} \quad (\forall x \in X)(x \in Ry \Rightarrow x \in A) \tag{2}$$

$$y \in R{\uparrow}A \quad \text{iff} \quad (\exists x \in X)(x \in Ry \wedge x \in A) \tag{3}$$

The underlying meaning is that $R{\downarrow}A$ is the set of elements *necessarily* belonging to C (strong membership), while $R{\uparrow}A$ is the set of elements *possibly* belonging to the concept C (weak membership); for y belongs to $R{\downarrow}A$ if all elements of X indistinguishable from y belong to A (hence there is no doubt that y also belongs to A), while y belongs to $R{\uparrow}A$ as soon as an element of A is indistinguishable from y. If y belongs to the boundary region $R{\uparrow}A \backslash R{\downarrow}A$, then there is doubt, because in this case y is at the same time indistinguishable from at least one element of A and at least one element of X that is not in A. We call (A_1, A_2) a rough set (in (X, R)) as soon as there is a set A in X such that $R{\downarrow}A = A_1$ and $R{\uparrow}A = A_2$ (see e.g. [14]).

Paraphrasing statements (2) and (3) and absorbing earlier suggestions in the same direction, the following definition of the lower and upper approximation of a fuzzy set A in X was given in [14], constructed by means of an implicator $\mathcal{I}$, a t–norm $\mathcal{T}$ and a fuzzy $\mathcal{T}$-equivalence relation R in X,

$$\begin{aligned} R{\downarrow}A(y) &= \inf_{x \in X} \mathcal{I}(R(x, y), A(x)) \\ R{\uparrow}A(y) &= \sup_{x \in X} \mathcal{T}(R(x, y), A(x)) \end{aligned}$$

for all y in X. In other words

$$\begin{aligned} R{\downarrow}A(y) &= \text{Incl}(Ry, A) \\ R{\uparrow}A(y) &= \text{Overl}(Ry \cap_{\mathcal{T}} A) \end{aligned}$$

(A_1, A_2) is called a fuzzy rough set (in (X, R)) as soon as there is a fuzzy set A in X such that $R{\downarrow}A = A_1$ and $R{\uparrow}A = A_2$. As we mentioned before however y may belong to different foresets to a given extent, not only to Ry. Therefore it appears natural to consider also the other foresets Rz to which y has a non–zero membership degree, and to assess the inclusion of Rz into A as well for the lower approximation, and the overlap of Rz and A for the upper approximation. Informally, this immediately results in the following (inexhaustive!) list of candidate definitions for the lower and the upper approximation of A:

1) y belongs to the lower approximation of A if
   a) all equivalence classes containing y are included in A
   b) at least one equivalence class containing y is included in A
   c) Ry is included in A
2) y belongs to the upper approximation of A if
   a) all equivalence classes containing y have a non-empty intersection with A
   b) at least one equivalence class containing y has a non-empty intersection with A
   c) Ry has a non-empty intersection with A

For A a crisp set and R a crisp relation, more formally we obtain the following definition.

Definition 6: Let R be a crisp relation and A a crisp set in X.

1) The tight, loose and (usual) lower approximation of A are defined as
   a) $y \in R{\downarrow}{\downarrow}A$ iff $(\forall z \in X)(y \in Rz \Rightarrow Rz \subseteq A)$
   b) $y \in R{\uparrow}{\downarrow}A$ iff $(\exists z \in X)(y \in Rz \wedge Rz \subseteq A)$
   c) $y \in R{\downarrow}A$ iff $Ry \subseteq A$

   for all y in X.
2) The tight, loose and (usual) upper approximation of A are defined as
   a) $y \in R{\downarrow}{\uparrow}A$ iff $(\forall z \in X)(y \in Rz \Rightarrow Rz \cap A \neq \emptyset)$
   b) $y \in R{\uparrow}{\uparrow}A$ iff $(\exists z \in X)(y \in Rz \wedge Rz \cap A \neq \emptyset)$
   c) $y \in R{\uparrow}A$ iff $Ry \cap A \neq \emptyset$

   for all y in X.

Option (c) corresponds to the well–known definition from the literature on rough set theory. In the crisp case options (1a) through (1c) coincide, as well as options (2a) through (2c) because then there is exactly one equivalence class to which y belongs, namely Ry. Paraphrasing these expressions for a fuzzy set and a fuzzy relation, we obtain the following definitions.

Definition 7: Let R be a fuzzy relation in X and A a fuzzy set in X.

1) The tight, loose and (usual) lower approximation of A are defined as
   a) $R{\downarrow}{\downarrow}A(y) = \inf_{z\in X} \mathcal{I}(Rz(y), \inf_{x\in X} \mathcal{I}(Rz(x), A(x)))$
   b) $R{\uparrow}{\downarrow}A(y) = \sup_{z\in X} \mathcal{T}(Rz(y), \inf_{x\in X} \mathcal{I}(Rz(x), A(x)))$
   c) $R{\downarrow}A(y) = \inf_{x\in X} \mathcal{I}(Ry(x), A(x))$

   for all y in X.
2) The tight, loose and (usual) upper approximation of A are defined as
   a) $R{\downarrow}{\uparrow}A(y) = \inf_{z\in X} \mathcal{I}(Rz(y), \sup_{x\in X} \mathcal{T}(Rz(x), A(x)))$
   b) $R{\uparrow}{\uparrow}A(y) = \sup_{z\in X} \mathcal{T}(Rz(y), \sup_{x\in X} \mathcal{T}(Rz(x), A(x)))$
   c) $R{\uparrow}A(y) = \sup_{x\in X} \mathcal{T}(Ry(x), A(x))$

   for all y in X.

The following proposition follows immediately from the definitions due to the symmetry of R.

Proposition 8: For every fuzzy set A in X

$$\begin{aligned} R{\downarrow}{\downarrow}A &= R{\downarrow}(R{\downarrow}A) \\ R{\uparrow}{\downarrow}A &= R{\uparrow}(R{\downarrow}A) \\ R{\downarrow}{\uparrow}A &= R{\downarrow}(R{\uparrow}A) \\ R{\uparrow}{\uparrow}A &= R{\uparrow}(R{\uparrow}A) \end{aligned}$$

C. Related Work

Although— to our knowledge—the tight and the loose lower and upper approximations have never been considered in the framework of fuzzy rough set theory, their crisp counterparts have already surfaced in classical rough set theory, albeit from different angles of interpretation. The first one is due to Cattaneo [3]. His approach to rough sets is remarkably different from others because it does not center around a notion of indistinguishability or similarity, but around a dual notion of discernibility. This discernibility is represented by a so-called preclusivity relation, which is an irreflexive and symmetrical relation. It can be obtained as the set theoretical complement $co\ R$ of an equivalence relation, or more generally of that of a tolerance relation R. Apart from the usual set–theoretical complement $co\ A$ of a set A, defined by

$$y \in co\ A \text{ iff } \neg(y \in A)$$

for all y in X, Cattaneo also defines the preclusive orthocomplement $R^{\#}(A)$ of A:

$$y \in R^{\#}(A) \text{ iff } (\forall x \in X)(x \in A \Rightarrow (x, y) \in co\ R) \tag{4}$$

$R^{\#}(A)$ is the set of elements that are discernible from all elements of A. Using also

$$R^b(A) = co(R^{\#}(co\ A))$$

Cattaneo introduces the $\mathcal{P}(X) - \mathcal{P}(X)$ mappings ν, $\mathbb{I}$, $\mathbb{C}$ and μ defined by

$$\begin{aligned} \nu(A) &= R^{\#}(co\ A) && \text{(necessity)} \\ \mathbb{I}(A) &= R^b(R^b(A)) && \text{(interior)} \\ \mathbb{C}(A) &= R^{\#}(R^{\#}(A)) && \text{(closure)} \\ \mu(A) &= co(R^{\#}(A)) && \text{(possibility)} \end{aligned}$$

for all A in $\mathcal{P}(X)$. Applying the law of contraposition ($p \Rightarrow q$ if and only if $\neg q \Rightarrow \neg p$) to formula (4) it is easy to see that

$$R^{\#}(A) = R{\downarrow}(co\ A)$$

Now for every crisp relation R and every crisp set A, $R{\uparrow}(co\ A) = co(R{\downarrow}A)$ and $R{\downarrow}(co\ A) = co(R{\uparrow}A)$ holds. This allows us to derive the following:

$$R^b(A) = co(R{\downarrow}A) = R{\uparrow}(co\ A)$$

and

$$\begin{aligned} \nu(A) &= R{\downarrow}A \\ \mathbb{I}(A) &= R{\uparrow}(R{\downarrow}A) \\ \mathbb{C}(A) &= R{\downarrow}(R{\uparrow}A) \\ \mu(A) &= R{\uparrow}A \end{aligned}$$

In [3] Cattaneo himself gives the full expressions of Definition 6, parts 1(c) and 2(c) for $\nu(A)$ and $\mu(A)$ respectively. In [8] $\mathbb{I}(A)$ and $\mathbb{C}(A)$ are linked to the expressions of Definition 6, parts 1(b) and 2(a) respectively, i.e. what we call tight lower approximation and loose upper approximation. In the crisp case, it makes sense to differentiate between $\nu(A)$ and $\mathbb{I}(A)$, and between $\mu(A)$ and $\mathbb{C}(A)$ if one is dealing with a tolerance relation R which is not an equivalence relation. In [8] it is also suggested to work with “tolerance classes of some iterations of tolerance relations”. It is not elaborated there what is meant by this, but let us assume that iteration refers to composition of relations which is usually defined as:

$$(x, z) \in R \circ S \text{ iff } (\exists y \in X)((x, y) \in R \wedge (y, z) \in S)$$

for R and S binary relations in X and x and z in X. The loose upper approximation and the tight lower approximation under R can be obtained as the usual upper and lower approximation under the composition of R with itself (for a symmetrical relation R), i.e.

$$\begin{aligned} (R \circ R){\uparrow}A &= R{\uparrow}(R{\uparrow}A) \\ (R \circ R){\downarrow}A &= R{\downarrow}(R{\downarrow}A) \end{aligned}$$

This kind of properties is closely related to associativity of compositions, because an operation such as taking the upper approximation of A under a relation R can be seen as a kind of composition. For a formal treatment of this connection we refer to [10].

IV. PROPOSITIONS

Throughout this section we will assume that R is a reflexive and symmetrical fuzzy relation in X, which are basic requirements if R is supposed to model similarity. The following proposition supports the idea of approximating a concept from the lower and the upper side (due to the reflexivity of R).

Proposition 9: [14] For every fuzzy set A in X

$$R{\downarrow}A \subseteq A \subseteq R{\uparrow}A$$

The lower and the upper approximation are monotonic operations due to the monotonicity of the fuzzy logical operators involved.

Proposition 10: [14] For every fuzzy set A and B in X

$$A \subseteq B \Rightarrow R{\downarrow}A \subseteq R{\downarrow}B$$

$$A \subseteq B \Rightarrow R{\uparrow}A \subseteq R{\uparrow}B$$

Applying Proposition 9 we conclude that the tight lower and the loose upper approximation are indeed a subset and a superset of A respectively (provided that R is reflexive of course).

Proposition 11: For every fuzzy set A in X

$$R{\downarrow}(R{\downarrow}A) \subseteq R{\downarrow}A \subseteq A \subseteq R{\uparrow}A \subseteq R{\uparrow}(R{\uparrow}A)$$

Note that in [4] it is suggested to use $R{\downarrow}(R{\downarrow}A)$, $R{\downarrow}A$, $R{\uparrow}A$ and $R{\uparrow}(R{\uparrow}A)$ as representations of the modified linguistic expressions extremely A, very A, more or less A and roughly A respectively (for R being a fuzzy relation modelling approximate equality). From Proposition 9 and Proposition 10 we obtain

$$R{\downarrow}A \subseteq R{\uparrow}(R{\downarrow}A) \subseteq R{\uparrow}A$$
$$R{\downarrow}A \subseteq R{\downarrow}(R{\uparrow}A) \subseteq R{\uparrow}A$$

for a reflexive fuzzy relation R, but no immediate information about a direct relationship between the loose lower and the tight upper approximation in terms of inclusion, and about how A itself fits in this picture. The following proposition sheds some light on this matter.

Proposition 12: [1] If R is a symmetrical fuzzy relation in X, $\mathcal{T}$ is a continuous t–norm and $\mathcal{I}$ its residual implicator then for every fuzzy set A in X

$$R{\uparrow}(R{\downarrow}A) \subseteq A \subseteq R{\downarrow}(R{\uparrow}A)$$

Proposition 12 does not hold in general for other choices of t–norms and implicators that do not fulfill the properties

$$\mathcal{T}(x, \mathcal{I}(x,y)) \leq y$$
$$y \leq \mathcal{I}(x, \mathcal{T}(x,y))$$

as Example 13 illustrates.

Example 13: Let X and R be defined as in Example 3 and let A be the fuzzy set in X defined as $A(a) = 1$ and $A(b) = 0.8$. Furthermore let $\mathcal{T} = \mathcal{T}_{\mathrm{M}}$ and $\mathcal{I} = \mathcal{I}_{\mathcal{T}_{\mathrm{M}},\mathcal{N}_s}$ be its S–implicator. Then $R{\uparrow}A(a) = 1$ and $R{\uparrow}A(b) = 0.8$, hence

$$R{\downarrow}(R{\uparrow}A)(a) = \min(\max(0,1), \max(0.8, 0.8)) = 0.8$$

which makes it clear that $A \not\subseteq R{\downarrow}(R{\uparrow}A)$.

To preserve the semantics of approximation from the lower and the upper side, we are therefore somewhat limited in our choice of fuzzy logical operators if we want to use the loose lower and the tight upper approximation. Furthermore if we assume that R is a fuzzy $\mathcal{T}$–equivalence relation, we obtain the following remarkable propositions.

Proposition 14: [1], [2], [14] If R is a fuzzy $\mathcal{T}$–equivalence relation in X, $\mathcal{T}$ is a continuous t–norm and $\mathcal{I}$ its residual implicator then for every fuzzy set A in X

1) $R{\uparrow}(R{\downarrow}A) = R{\downarrow}A$
2) $R{\downarrow}(R{\uparrow}A) = R{\uparrow}A$

Proposition 15: [14] If R is a fuzzy $\mathcal{T}$–equivalence relation in X, $\mathcal{T}$ is a continuous t–norm and $\mathcal{I}$ is a continuous implicator such that $\mathcal{T}$ and $\mathcal{I}$ satisfy the weak shunting principle, then for every fuzzy set A in X

1) $R{\uparrow}(R{\uparrow}A) = R{\uparrow}A$
2) $R{\downarrow}(R{\downarrow}A) = R{\downarrow}A$

The weak shunting principle is only needed in the proof of part 2 of Proposition 15 to link the implicator $\mathcal{I}$ that appears in the definition of the lower approximation with the t–norm $\mathcal{T}$ for which R is $\mathcal{T}$-transitive.

Giving up on $\mathcal{T}$–transitivity means giving up on these propositions, as Example 16 shows.

Example 16: Let $X = [0,1]$ and A the fuzzy set in X defined as $A(x) = x$, for all x in X. Let the reflexive and symmetrical fuzzy relation R in X be defined as

$$R(x,y) = \begin{cases} 1 & \text{if } |x-y| < 0.1 \\ 0 & \text{otherwise} \end{cases}$$

for all x and y in X. One can verify that for y in X:

$$\begin{aligned} R{\downarrow}A(y) &= \inf_{z \in X} \mathcal{I}(R(z,y), A(z)) \\ &= \inf\{z \mid z \in X \wedge z \in]y-0.1, y+0.1[\} \\ &= \max(0, y-0.1) \end{aligned}$$

Hence $R{\downarrow}A(1) = 0.9$. Furthermore

$$\begin{aligned} R{\downarrow}(R{\downarrow}A)(1) &= \inf_{z \in X} \mathcal{I}(R(z,1), \max(0, z-0.1)) \\ &= \inf\{\max(0, z-0.1) \mid z \in]0.9, 1]\} = 0.8 \end{aligned}$$

which shows that $R{\downarrow}(R{\downarrow}A) \neq R{\downarrow}A$. Analogously

$$\begin{aligned} R{\uparrow}A(y) &= \sup_{z \in X} \mathcal{T}(A(z), R(z,y)) \\ &= \max\{z \mid z \in X \wedge z \in]y-0.1, y+0.1[\} \\ &= \min(1, y+0.1) \end{aligned}$$

hence $R{\uparrow}A(0) = 0.1$. Furthermore

$$\begin{aligned} R{\uparrow}(R{\uparrow}A)(0) &= \sup_{z \in X} \mathcal{T}(\min(1, z+0.1), R(z,0)) \\ &= \sup\{\min(1, z+0.1) \mid z \in [0, 0.1[\} = 0.2 \end{aligned}$$

so $R{\uparrow}(R{\uparrow}A) \neq R{\uparrow}A$.

V. Concluding Remarks

Exploiting the truly fuzzy characteristic that an element can belong (to some degree) to different sets at the same time, we have detected that traditional approaches to fuzzy rough set theory overlooked a step in the fuzzification process. Under the conditions of Proposition 14 and 15 however, the tight, the loose and the (usual) upper approximation coincide, and so do the tight, the loose and the (usual) lower approximation. The observation that the consideration of other foresets than Ry does not bring anything new if we are working with a fuzzy $\mathcal{T}$-equivalence relation and some nice fuzzy logical operators can either be reasuring: "although we did not consider this option in the past, no changes are required in our model, which shows that it is solid", but it can also be disturbing: "why does taking into account a property which is so fundamental to fuzzy set theory has so little impact on our model?" In the crisp case all of the equalities in Proposition 14 and 15 hold because the equivalence classes of a crisp equivalence relation either coincide or are disjoint. As a result each element y of X belongs to exactly one equivalence class, i.e. $Ry(y) = 1$ and $Rz(y) = 0$ for all z in X such that $z \neq y$. If R is a fuzzy $\mathcal{T}$–equivalence relation, in general we can no longer make this kind of straightforward verification. However in this case the interplay between the nice fuzzy logical operators and the $\mathcal{T}$–transitivity of R takes care of things.

Example 16 illustrated that the usual approximations and the newly introduced one may cease to coincide if the fuzzy relation R is not required to be $\mathcal{T}$–transitive. Since in the framework of fuzzy rough sets, R models approximate equality of objects w.r.t their attribute values, and the suitability of the requirement of $\mathcal{T}$–transitivity for these kinds of fuzzy relations can be questioned (to say the least) because of their incompatibility with the Poincaré paradox, this certainly points us towards an interesting direction for future research.

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