The Compositional Rule of Inference in an Intuitionistic Fuzzy Logic Setting

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Abstract. The incorporation of imprecise, linguistic information into logical deduction processes, as opposed to the practice of traditional two–valued propositional logic and set theory, continues to be a predominant feature of fuzzy expert systems. Throughout the literature, we can find all sorts of intelligent inference schemes acting under imprecision; common to most approaches is their reliance on if–then rules of the kind "IF X is A THEN Y is B", where A and B are fuzzy sets in given universes U and V. Intuitively, fuzzy sets (FSs) can be used to model elastic constraints on the values a variable may assume. While the theory of FS–based approximate reasoning is surely a well–established and commonly applied one, there is still a demand for further expanding the expressiveness of the formalism. One such improvement can be obtained by using Atanassov's [1] intuitionistic fuzzy sets (IFSs), of which FSs are specific instances, and which highlight the fundamental importance of negation: the degree to which a proposition is false, or equivalently to which an object does not belong to a set, is given an independent status here. In this paper we will contribute to the further development of this relatively young theory, by generalizing the well-known Compositional Rule of Inference (CRI) to IFSs. We also deal with the related problem of checking the validity of the inference, as motivated in [3].

1 Introduction and preliminaries

Inference is defined as a procedure for deducing new facts out of existing ones on the basis of formal deduction rules. Classical paradigms like two–valued propositional and predicate logic, exhibit some important drawbacks (lack of expressivity in describing incomplete and/or imprecise knowledge, high computational complexity) that make them unsuitable for application in automated deduction systems (e.g. for medical diagnosis). To alleviate these difficulties, Zadeh in 1973 introduced a formalism called approximate reasoning to cope with problems which are too complex for exact solution but which do not require a high degree of precision. [12] His work is centered around the notions of a fuzzy set and a fuzzy restriction.

In his seminal 1965 paper [11], Zadeh generalized ordinary sets to fuzzy sets (FSs, for short), allowing an element $u \in U$ to belong to any degree of membership in [0,1] (denoted $A(u)$) to a fuzzy set A in U. It is clear that the extension equivalently gives rise to a continuum of truth values between 0 and 1 for a logical proposition P.

Definition 1.1 (Fuzzy set) *A fuzzy set A in a given universe U is a mapping from U into the unit interval $[0,1]$. The class of fuzzy sets in U is denoted $\mathcal{F}(U)$.*

To define the intersection and union of fuzzy sets (equivalently, conjunction and disjunction of fuzzy propositions), so–called t–norms and t–conorms are used: a t–norm is any symmetric, associative, increasing $[0,1] \times [0,1] \to [0,1]$ mapping T satisfying $T(1,x) = x$ for every $x \in [0,1]$, whereas for a t–conorm S the last property is replaced by $S(0,x) = x$ for every $x \in [0,1]$. t–norms give rise to fuzzy intersections, in the sense that $A \cap_T B(u) = T(A(u), B(u))$ for every $u \in U$ and T a t–norm. An analogous result holds of course for t–conorms and unions.

Now consider the statement: "Paul is very old". Modelling "very old" as a fuzzy set on a suitable range of ages, this statement constitutes a so–called fuzzy restriction on the possible values of Paul's age rather than an assertion about the membership of Paul in a class of individuals. [12] From a logical perspective, it is interesting to see how people are able to combine such information efficiently in a Modus Ponens–like fashion to allow for inferences of the following kind:

IF	bath water is "too hot"	THEN	I'm apt to get burnt
	bath water is "really rather hot"		
			I'm quite apt to get burnt

The technique used above is in fact less restrictive than the actual MP from propositional logic since it doesn't require the observed fact ("really rather hot") and the antecedent of the rule ("too hot") to coincide to yield a meaningful conclusion. The need emerges for a flexible, qualitative scale of measuring to what extent the antecedent is fulfilled, on the basis of which we could obtain an approximate idea (stated under the form of another fuzzy restriction) of the value of the consequent variable.

With his introduction of a calculus of fuzzy restrictions [12], Zadeh paved the way towards a reasoning scheme called Generalized Modus Ponens (GMP) to systematize deductions like the example we presented. Since his pioneering work, many researchers have sought for efficient realizations[1] of

[1] By "realization", we mean any computational procedure unambiguously defining the output in terms of the inputs

this approximate inference scheme. In section 2 we will formally define the GMP and survey its most common realization, the Compositional Rule of Inference (CRI). Section 3 introduces the notion of Intuitionistic Fuzzy Sets (IFSs) and their connectives. In section 4, we proceed to extend the CRI to the IFS setting. In section 5, we address a common validation procedure based on the notion of Intuitionistic Fuzzy Tautology (IFT) and discuss how it affects our reasoning processes. Finally, section 6 offers some options for future research.

2 FS–based Compositional Rule of Inference

We start by recalling from [7] the definition of the main concept that we are concerned with:

Definition 2.1 (Generalized Modus Ponens, GMP) *Let X and Y be variables assuming values in U, resp. V. Consider then a fuzzy rule "IF X is A, THEN Y is B" and a fuzzy fact (or observation) "X is A'" ($A, A' \in \mathcal{F}(U), B \in \mathcal{F}(V)$). The GMP allows deduction of a fuzzy fact "Y is B'", with $B' \in \mathcal{F}(V)$.*

Expressing this under the form of an inference scheme, we get:

$$\begin{array}{ll} \textit{IF } X \textit{ is } A, \textit{ THEN} & Y \textit{ is } B \\ X \textit{ is } A' & \\ \hline & Y \textit{ is } B' \end{array}$$

Definition 2.1 does not state what the fuzzy restriction B' should be when A, A' and B are given. A lot of approaches have been proposed for this purpose ([6], among others, gives a survey), the most common one relying on the so–called Compositional Rule of Inference, a convenient mechanism for calculating with fuzzy restrictions introduced by Zadeh in [12].

Definition 2.2 (Compositional Rule of Inference, CRI) *[7] Let X and Y be defined as in definition 2.1. Consider also fuzzy facts "X is A'" and "X and Y are R", where $A' \in \mathcal{F}(U), R \in \mathcal{F}(U \times V)$ (R is a fuzzy relation between U and V). The CRI allows us to infer the fuzzy fact: "Y is $R \circ_T A'$", in which the direct image of A' under R, denoted[2] $R \circ_T A'$, is defined as, for $v \in V$:*

$$R \circ_T A'(v) = \sup_{u \in U} T(A'(u), R(u, v))$$

Expressing this under the form of an inference scheme, we get:

[2]Some people prefer to speak of the "composition of R with A'" hence the appearance of the composition symbol.

$$\begin{array}{l} X \ \textit{is} \ A' \\ X \ \textit{and} \ Y \ \textit{are} \ R \\ \hline Y \ \textit{is} \ R \circ_T A' \end{array}$$

For definition 2.2 to be a realization of the GMP, R must be a relational representation of a fuzzy implicator, an extension of the classical implication operator:

Definition 2.3 (Fuzzy implicator) *[10] A fuzzy implicator is any $[0,1]^2 \to [0,1]$ mapping $\mathcal{I}$ for which the restriction to $\{0,1\}^2$ coincides with classical implication: $\mathcal{I}(0,0)=1$, $\mathcal{I}(1,0)=0$, $\mathcal{I}(0,1)=1$, $\mathcal{I}(1,1)=1$. Moreover, $\mathcal{I}$ should satisfy the following monotonicity criteria:*

$$(\forall y \in [0,1])(\forall (x,x') \in [0,1]^2)(x \leq x' \Rightarrow \mathcal{I}(x,y) \geq \mathcal{I}(x',y)) \tag{1.1}$$

$$(\forall x \in [0,1])(\forall (y,y') \in [0,1]^2)(y \leq y' \Rightarrow \mathcal{I}(x,y) \leq \mathcal{I}(x,y')) \tag{1.2}$$

Given $\mathcal{I}$ and A and B, the fuzzy sets used in definition 2.1, R is defined as, for $(u,v) \in U \times V$: $R(u,v) = \mathcal{I}(A(u), B(v))$. The two most important classes of fuzzy implicators are called $\mathcal{S}$– and $\mathcal{R}$–implicators, and are defined as follows[3]:

Definition 2.4 ($\mathcal{S}$–implicator) *[5] Let S be a t–conorm. The $\mathcal{S}$–implicator generated by S is the mapping $\mathcal{I}_S$ defined as:*

$$\begin{array}{rccl} \mathcal{I}_S: & [0,1]^2 & \to & [0,1] \\ & (x,y) & \mapsto & S(1-x,y), \quad \forall (x,y) \in [0,1]^2 \end{array}$$

Definition 2.5 ($\mathcal{R}$–implicator) *[5] Let T be a t–norm. The $\mathcal{R}$–implicator generated by T is the mapping $\mathcal{I}_T$ defined as:*

$$\begin{array}{rccl} \mathcal{I}_T: & [0,1]^2 & \to & [0,1] \\ & (x,y) & \mapsto & \sup\{\gamma \in [0,1] | T(x,\gamma) \leq y\}, \quad \forall (x,y) \in [0,1]^2 \end{array}$$

3 Intuitionistic Fuzzy Sets

IFSs, first introduced by Atanassov [1] in 1983, generalize Zadeh's fuzzy sets. While FSs merely give the degree of membership of an element in a set, IFSs also involve a degree of non–membership.

Definition 3.1 *An intuitionistic fuzzy set in a universe U is any object A of the form $A = \{(u, \mu_A(u), \nu_A(u)) | u \in U\}$, where the membership function μ_A and the non–membership function ν_A are $U \to [0,1]$ mappings satisfying $(\forall u \in U)(\mu_A(u) + \nu_A(u) \leq 1)$. The class of all IFSs in U is denoted $\mathcal{IF}(U)$.*

[3] It is easily verified that they are indeed fuzzy implicators. [6]

Clearly any FS $A \in \mathcal{F}(U)$ has an IFS representation where for any $u \in U$ the degree of non–membership equals one minus the degree of membership. There also exist straightforward extensions of the FS union and intersection to IFSs. Let T be a t-norm and S a t-conorm. Then the generalized intersection $A \cap_{T,S} B$ of two IFSs A and B in U can be defined as $A \cap_{T,S} B = \{(u, T(\mu_A(u), \mu_B(u)), S(\nu_A(u), \nu_B(u))) | u \in U\}$. The resulting object is again an IFS provided $T \leq S^*$, where S^* denotes the dual t-norm of S, defined as $S^*(x,y) = 1 - S(1-x, 1-y)$ for all x and y in $[0,1]$. Indeed, from $T \leq S^*$ and using the increasing property of the t-conorm S, we obtain $T(\mu_A(u), \mu_B(u)) \leq 1 - S(1-\mu_A(u), 1-\mu_B(u)) \leq 1 - S(\nu_A(u), \nu_B(u))$. Putting $\nu_A(u) = 1 - \mu_A(u)$ and $\nu_B(u) = 1 - \mu_B(u)$, it is clear that the condition $T \leq S^*$ is also necessary. A similar result can be obtained for the IFS union $\cup_{S,T}$, under the condition $S \leq T^*$, which is equivalent to $T \leq S^*$.

4 IFS–based Compositional Rule of Inference

As discussed in the previous section, IFSs offer a more general framework than FSs do, thus allowing representation of relations between variables that could previously not be described (see e.g. [2] [8] for some real–world examples). Also in [2], a first attempt is made to endow fuzzy expert systems with concepts from IFS theory, indicating the real interest excited by these structures. It would therefore be nice to find some suitable IFS adaptation of the CRI, by far the most common means of deduction in the FS setting.

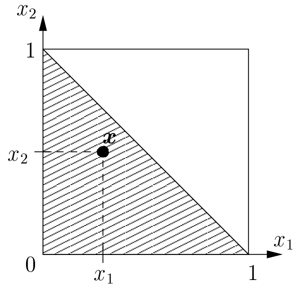
Before we can generalize the GMP and the CRI, we have to introduce some preliminary concepts.

Definition 4.1 (Lattice $(L^*, \leq_{L^*})$) *Define a lattice $(L^*, \leq_{L^*})$ such that:*

$$L^* = \{(x_1, x_2) \in [0,1]^2 \mid x_1 + x_2 \leq 1\}$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \wedge x_2 \geq y_2$$

The shaded area in the figure is the set of elements $x = (x_1, x_2)$ belonging to L^*.



The lattice $(L^*, \leq_{L^*})$ is a complete lattice: for each $A \subseteq L^*$,

$$\begin{array}{rcl} \sup A & = & (\sup\{x_1 \in [0,1] \mid (\exists x_2 \in [0,1])((x_1,x_2) \in A)\}, \\ & & \inf\{x_2 \in [0,1] \mid (\exists x_1 \in [0,1])((x_1,x_2) \in A)\})\,, \\ \inf A & = & (\inf\{x_1 \in [0,1] \mid (\exists x_2 \in [0,1])((x_1,x_2) \in A)\}, \\ & & \sup\{x_2 \in [0,1] \mid (\exists x_1 \in [0,1])((x_1,x_2) \in A)\})\,. \end{array}$$

Equivalently, this lattice can also be defined as an algebraic structure $(L^*, \wedge, \vee)$ where the meet operator $\wedge$ and the join operator $\vee$ are defined as follows, for $(x_1, x_2), (y_1, y_2) \in L^*$:

$$(x_1, x_2) \wedge (y_1, y_2) = (\min(x_1, y_1), (\max(x_2, y_2))$$

$$(x_1, x_2) \vee (y_1, y_2) = (\max(x_1, y_1), \min(x_2, y_2))$$

For our purposes, we will also consider a generalized meet operator $\wedge_{T,S}$ and join operator $\vee_{S,T}$ on $(L^*, \leq_{L^*})$:

$$(x_1, x_2) \wedge_{T,S} (y_1, y_2) = (T(x_1, y_1), S(x_2, y_2))$$

$$(x_1, x_2) \vee_{S,T} (y_1, y_2) = (S(x_1, y_1), T(x_2, y_2))$$

for given t–norm T and t–conorm S satisfying $T \leq S^*$. Again, it can be shown that the condition $T \leq S^*$ is necessary and sufficient for these operators to be well-defined.

Lastly, we define an order–reversing mapping $\mathcal{N}$ by $\mathcal{N}(x_1, x_2) = (x_2, x_1)$, $\forall (x_1, x_2) \in L^*$.

We now propose the following extensions for intuitionistic fuzzy implicators, as well as the special instances of $\mathcal{S}$– and $\mathcal{R}$–implicators.

Definition 4.2 (Intuitionistic Fuzzy Implicator) *An intuitionistic fuzzy implicator is any $(L^*)^2 \to L^*$–mapping $\mathcal{I}$ satisfying the border conditions*

$$\begin{array}{rcl} \mathcal{I}(0_{L^*}, 0_{L^*}) & = & 1_{L^*}, \\ \mathcal{I}(1_{L^*}, 0_{L^*}) & = & 0_{L^*}, \\ \mathcal{I}(0_{L^*}, 1_{L^*}) & = & 1_{L^*}, \\ \mathcal{I}(1_{L^*}, 1_{L^*}) & = & 1_{L^*}, \end{array}$$

where $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$ are the identities of $(L^*, \leq_{L^*})$. Moreover we require $\mathcal{I}$ to be decreasing in its first, and increasing in its second component, i.e.

$$(\forall y \in L^*)(\forall (x, x') \in (L^*)^2)(x \leq_{L^*} x' \Rightarrow \mathcal{I}(x,y) \geq_{L^*} \mathcal{I}(x', y)) \qquad (1.3)$$

$$(\forall x \in L^*)(\forall (y, y') \in (L^*)^2)(y \leq_{L^*} y' \Rightarrow \mathcal{I}(x,y) \leq_{L^*} \mathcal{I}(x, y')) \qquad (1.4)$$

Definition 4.3 ($\mathcal{S}$–implicator) *Let T be a t-norm, S a t-conorm satisfying $T \leq S^*$, and $\mathcal{N}$ an involutive[4] order–reversing operator on L^*. The $\mathcal{S}$–implicator generated by T, S and $\mathcal{N}$ is the mapping $\mathcal{I}_{S,T,\mathcal{N}}$ defined as, for $(x, y) \in (L^*)^2$:*

$$\mathcal{I}_{S,T,\mathcal{N}}(x, y) = \mathcal{N}(x) \vee_{S,T} y$$

Definition 4.4 ($\mathcal{R}$–implicator) *Let T be a t-norm, S a t-conorm satisfying $T \leq S^*$. The $\mathcal{R}$–implicator generated by T and S is the mapping $\mathcal{I}_{T,S}$ defined as, for $(x, y) \in (L^*)^2$:*

$$\mathcal{I}_{T,S}(x, y) = \sup\{\gamma \in L^* \mid x \wedge_{T,S} \gamma \leq_{L^*} y\}$$

Theorem 4.1 *The mappings $\mathcal{I}_{S,T,\mathcal{N}}$ and $\mathcal{I}_{T,S}$ are intuitionistic fuzzy implicators.*

Proof. It is easy to verify that the defined operators satisfy the border conditions. We now prove that they satisfy the hybrid monotonicity properties.

(i) Since each t-norm T and each t-conorm S are increasing in both components, it can easily be seen that, for all $(a_1, a_2), (b_1, b_2), (c_1, c_2)$, $(d_1, d_2) \in L^*$, $(a_1, a_2) \leq_{L^*} (b_1, b_2)$ and $(c_1, c_2) \leq_{L^*} (d_1, d_2)$ implies $(S(a_1, c_1), T(a_2, c_2)) \leq_{L^*} (S(b_1, d_1), T(b_2, d_2))$, i.e. $(a_1, a_2) \vee_{S,T} (c_1, c_2)$ $\leq_{L^*} (b_1, b_2) \vee_{S,T} (d_1, d_2)$. Hence $\vee_{S,T}$ is increasing in both components. Similarly $\wedge_{T,S}$ is increasing in both components.

Since $\mathcal{N}$ is order–reversing, we obtain successively, for $x, x', y \in L^*$,

$$\begin{array}{rcl} x & \leq_{L^*} & x' \\ \mathcal{N}(x) & \geq_{L^*} & \mathcal{N}(x') \\ \mathcal{N}(x) \vee_{S,T} y & \geq_{L^*} & \mathcal{N}(x') \vee_{S,T} y \\ \mathcal{I}_{S,T,\mathcal{N}}(x, y) & \geq_{L^*} & \mathcal{I}_{S,T,\mathcal{N}}(x', y) \end{array}$$

It is equally obvious that $\mathcal{I}_{S,T,\mathcal{N}}$ is increasing in its second component.

(ii) Now we prove that an arbitrary $\mathcal{R}$–implicator $\mathcal{I}_{T,S}$ is hybrid monotonous. Let $x = (x_1, x_2), x' = (x'_1, x'_2), y = (y_1, y_2) \in L^*$ such that $x \leq_{L^*} x'$. Then

$$\begin{aligned} &\{(\gamma_1, \gamma_2) \in L^* \mid (T(x_1, \gamma_1), S(x_2, \gamma_2)) \leq_{L^*} (y_1, y_2)\} \\ &\qquad \supseteq \{(\gamma_1, \gamma_2) \in L^* \mid (T(x'_1, \gamma_1), S(x'_2, \gamma_2)) \leq_{L^*} (y_1, y_2)\} \end{aligned}$$

since $T(x_1, \gamma_1) \leq T(x'_1, \gamma_1)$ and $S(x_2, \gamma_2) \geq S(x'_2, \gamma_2)$. Hence

$$\begin{aligned} &\sup\{(\gamma_1, \gamma_2) \in L^* \mid (T(x_1, \gamma_1), S(x_2, \gamma_2)) \leq_{L^*} (y_1, y_2)\} \\ &\qquad \geq \sup\{(\gamma_1, \gamma_2) \in L^* \mid (T(x'_1, \gamma_1), S(x'_2, \gamma_2)) \leq_{L^*} (y_1, y_2)\} \end{aligned}$$

Analogously, monotonicity in the second component is obtained.

[4] An $X \to X$–mapping f is involutive iff, for all $x \in X$, $f(f(x)) = x$

$\square$

A generalization of the CRI will require us to define the direct image of an IFS under an intuitionistic fuzzy relation (IFR).

Definition 4.5 *Let $A \in \mathcal{IF}(U)$, $R \in \mathcal{IF}(U \times V)$, T a t–norm and S a t–conorm satisfying $T \leq S^*$. The direct image $R \circ_{T,S} A$ is defined as:*

$$R \circ_{T,S} A = \left\{ \left(v, \sup_{u \in U} T(\mu_A(u), \mu_R(u,v)), \inf_{u \in U} S(\nu_A(u), \nu_R(u,v)) \right) \mid v \in V \right\} \tag{1.5}$$

Now we have all the necessary tools to generalize the GMP and the CRI. Definition 2.1 can be maintained if we replace the word "fuzzy" by "intuitionistic fuzzy", and $\mathcal{F}(U)$ and $\mathcal{F}(V)$ by $\mathcal{IF}(U)$ and $\mathcal{IF}(V)$ respectively. We call this pattern Intuitionistic Generalized Modus Ponens (IGMP).

Just as in the fuzzy case, a realization of the IGMP can be obtained by defining the output B' as the direct image of the input A' under an intuitionistic fuzzy relation R representing the intuitionistic fuzzy rule. This gives rise to the following generalization of the CRI.

Definition 4.6 (IFS-based Compositional Rule of Inference, ICRI) *Let X and Y be variables assuming values in U, resp. V. Consider intuitionistic fuzzy facts "X is A'" and "X and Y are R", where $A' \in \mathcal{IF}(U), R \in \mathcal{IF}(U \times V)$ (R is an intuitionistic fuzzy relation between U and V). The ICRI allows us to infer the fuzzy fact: "Y is $B' = R \circ_{T,S} A'$", where (T, S) is a given pair of a t–norm and a t–conorm satisfying $T \leq S^*$.*
Expressing this under the form of an inference scheme, we get:

$$\begin{array}{l} X \textit{ is } A' \\ (X,Y) \textit{ is } R \\ \hline Y \textit{ is } B' = R \circ_{T,S} A' \end{array}$$

We use an IF implicator $\mathcal{I}$ to define R. Given IFSs A and B and $\mathcal{I}$, we calculate, for $(u,v) \in U \times V$,

$$(\mu_R(u,v), \nu_R(u,v)) = \mathcal{I}((\mu_A(u), \nu_A(u)), (\mu_B(v), \nu_B(v))),$$

thus defining R. Using this definition, it is clear that the ICRI is an extension of the fuzzy-based CRI.

5 Validity of the Modus Ponens

As pointed out in [3], validity of the modus ponens (MP) is essential if one is interested in passing from hypothesis to conclusions without loss in the degree of truth. Before we introduce the definition of validity, we first give the following definition.

Definition 5.1 (Intuitionistic fuzzy tautology, IFT) *[3] Let $a = (a_1, a_2) \in L^*$, then a is said to be an "intuitionistic fuzzy tautology" if and only if $a_1 \geq a_2$.*

Definition 5.2 (Validity of the modus ponens) *[3] We say that the MP is valid for an IF implicator $\mathcal{I}$ iff, for $a = (a_1, a_2), b = (b_1, b_2) \in L^*$, we have*[5]

$$a_1 \geq a_2 \wedge pr_1(\mathcal{I}(a, b)) \geq pr_2(\mathcal{I}(a, b)) \Rightarrow b_1 \geq b_2$$

This amounts to: "if a is an IFT and $\mathcal{I}(a, b)$ is an IFT, then b is an IFT".

Unfortunately, the implicators defined in the previous section, do not satisfy the validity of the modus ponens, as shown in the following theorem.

Theorem 5.1

- *If for the mapping $\mathcal{N}$ there exists an $x = (x_1, x_2) \in L^*$ such that $x_1 \geq x_2$ and $pr_1(\mathcal{N}(x)) \neq 0$, then the modus ponens in not valid for the $\mathcal{S}$–implicator $\mathcal{I}_{S,T,\mathcal{N}}$.*
- *The modus ponens is not valid for any $\mathcal{R}$–implicator.*

Proof.

- Let $x = (x_1, x_2) \in L^*$ with $x_1 \geq x_2$ be such that $\mathcal{N}(x) = (x'_1, x'_2)$ with $x'_1 \neq 0$, and let $y = (0, x'_1)$. Then

  $$\mathcal{I}_{S,T,\mathcal{N}}(x, y) = \mathcal{N}(x) \vee_{S,T} y = (S(x'_1, 0), T(x'_2, x'_1)) = (x'_1, T(x'_2, x'_1)),$$

  with $x'_1 \geq T(x'_2, x'_1)$. This shows that the modus ponens is not valid for $\mathcal{I}_{S,T,\mathcal{N}}$.

- Let $x = (x_1, x_2) \in L^*$ and $y = (y_1, y_2) \in L^*$ such that $y_2 > y_1 \geq x_1 \geq x_2 = 0$. Then $T(x_1, \gamma_1) \leq x_1 \leq y_1$, $\forall \gamma_1 \in [0, 1]$ and $S(x_2, \gamma_2) = \gamma_2$, hence

  $$\sup\{(\gamma_1, \gamma_2) \in L^* \mid (T(x_1, \gamma_1), S(x_2, \gamma_2)) \leq_{L^*} y\} = (1 - y_2, y_2).$$

  If $y_2 \leq \frac{1}{2}$, then $1 - y_2 \geq y_2$. For the MP to be valid, we need:

  $$\begin{aligned}&(\forall(a_1, a_2) \in L^*)(\forall(b_1, b_2) \in L^*)(a_1 \geq a_2 \wedge \\ &pr_1(\mathcal{I}((a_1, a_2), (b_1, b_2))) \geq pr_2(\mathcal{I}((a_1, a_2), (b_1, b_2))) \Rightarrow b_1 \geq b_2)\end{aligned}$$

  The chosen x and y satisfy $x_1 \geq x_2$ and $pr_1(\mathcal{I}(x, y)) \geq pr_2(\mathcal{I}(x, y))$, but have $y_1 < y_2$. Hence the MP is not valid.

[5]The projection mappings pr_1, pr_2 are defined, for any $(x_1, x_2) \in L^*$, as: $pr_1(x_1, x_2) = x_1, pr_2(x_1, x_2) = x_2$

$\square$

In [3] Atanassov defines a number of alleged implicators for which the modus ponens is valid. Unfortunately, none of his proposed mappings is an IF implicators in the sense of definition 4.2 (either the border conditions, or the hybrid monotonicity, or both, are violated).

One may wrongly conclude from the above discussion that the implicators defined in section 4 are of minor interest. Indeed, in the fuzzy case validity is defined as, for $a, b \in [0,1]$ and $\mathcal{I}$ a fuzzy implicator, "a is a fuzzy tautology (FT) and $\mathcal{I}(a,b)$ is a FT implies b is a FT", where $x \in [0,1]$ is said to be a FT if and only if $x \geq \frac{1}{2}$ (cfr. [3]). The modus ponens is not valid for fuzzy $\mathcal{S}$–implicators either.[6] However, most of the commonly used fuzzy implicators belong to this class.

Set apart from these considerations, we still find it useful to look for intuitionistic fuzzy implicators for which the modus ponens does hold. This will be the subject of a future paper.

6 Conclusion and Future Work

The interest for Intuitionistic Fuzzy Sets from the perspective of logical deduction will continue to grow as more people become familiar with their straightforward semantics and flexible operations. As it turns out, we could find very useful results not previously established, notably the extension of a very wide range of fuzzy implicators to IFS's and their application in the intuitionistic CRI. The consistency of the reached approach still needs to be looked into systematically. The procedure of checking for validity lined out in section 5, along with several other criteria that the inference should satisfy, provides us with various yardsticks for evaluating the performance of our inference scheme. For FS–based GMP, extensive studies have been carried out for this purpose [4] [9].

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