On the representation of intuitionistic fuzzy t-norms and t-conorms

Glad Deschrijver, Chris Cornelis, Etienne E. Kerre

Department of Mathematics and Computer Science, Ghent University
Fuzziness and Uncertainty Modelling Research Unit
Krijgslaan 281 (S9), B-9000 Gent, Belgium
E-mail: {glad.deschrijver|chris.cornelis|etienne.kerre}@rug.ac.be
Homepage: http://fuzzy.rug.ac.be

Abstract

In fuzzy set theory, an important class of triangular norms and conorms is the class of continuous Archimedean nilpotent triangular norms and conorms. It has been shown that for such t-norms T there exists a bijection φ on $[0,1]$ such that T is the φ-transform of the Łukasiewicz t-norm. From this class of t-norms an important class of fuzzy implicators can be generated : the class of Łukasiewicz implicators. In this paper we introduce the notion of intuitionistic fuzzy t-norm and t-conorm, and investigate under which conditions a similar representation theorem can be obtained. We also establish a representation theorem for intuitionistic fuzzy Łukasiewicz implicators.

Keywords: intuitionistic fuzzy set, intuitionistic fuzzy triangular norm and conorm, representation theorem, archimedean property, nilpotency, φ–transform

1 Introduction

An important notion in fuzzy set theory is that of triangular norms and conorms : t-norms and t-conorms are used to define a generalized union and intersection of fuzzy sets, to extend the composition of fuzzy relations e.g. for use in approximate reasoning systems, and for many other purposes. Another important operation is the fuzzy implicator. With the interest in intuitionistic fuzzy sets in general, and intuitionistic fuzzy relational calculus in particular (see e.g. [5, 6, 9, 10, 11]) quickly expanding, it is definitely worthwhile to extend these important connectives to intuitionistic fuzzy set theory, and apply the same formal investigation to them as was done in the fuzzy set case.

In fuzzy set theory continuous, Archimedean, nilpotent t-norms play a very important role (see e.g. [17]); they occur for instance in the theory of Łukasiewicz implicators, i.e. fuzzy implicators that fulfill the entire axiom set of Smets and Magrez [19]. A representation theorem was established for continuous Archimedean nilpotent t-norms : a t-norm T is continuous, Archimedean and nilpotent if and only if there exists a bijection φ on $[0,1]$ such that T is the

φ-transform of the Łukasiewicz t-norm T_W, i.e. $T = \varphi^{-1} \circ T_W \circ (\varphi, \varphi)$, where T_W is defined as $T_W(x, y) = \max(0, x + y - 1)$, for all $x, y \in [0, 1]$. An analogous result holds for t-conorms, and from these results a representation of fuzzy Łukasiewicz implicators is deduced. In this paper we extend the notions of t-norm and t-conorm to the intuitionistic fuzzy case, and we generalize said representation theorems to these intuitionistic fuzzy connectives.

2 Intuitionistic fuzzy sets

Intuitionistic fuzzy sets were introduced by Atanassov in 1983 and are defined as follows.

Definition 2.1 [1, 2, 3] *An intuitionistic fuzzy set A in a universe U is an object*

$$A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\},$$

where, for all $u \in U$, $\mu_A(u) \in [0, 1]$ and $\nu_A(u) \in [0, 1]$ are called the membership degree and the non-membership degree, respectively, of u in A, and furthermore satisfy $\mu_A(u) + \nu_A(u) \leq 1$.

Deschrijver and Kerre [13] have shown that intuitionistic fuzzy sets can also be seen as L-fuzzy sets in the sense of Goguen [15]. Consider the set L^* and operation $\leq_{L^*}$ defined by :

$$L^* = \{(x_1, x_2) \mid (x_1, x_2) \in [0, 1]^2 \wedge x_1 + x_2 \leq 1\},$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \wedge x_2 \geq y_2, \quad \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ is a complete lattice [13]. For each $A \subseteq L^*$ we have

$$\begin{aligned}\sup A = (&\sup\{x_1 \mid x_1 \in [0, 1] \wedge (\exists x_2 \in [0, 1 - x_1])((x_1, x_2) \in A)\},\\ &\inf\{x_2 \mid x_2 \in [0, 1] \wedge (\exists x_1 \in [0, 1 - x_2])((x_1, x_2) \in A\}),\\ \inf A \ = (&\inf\{x_1 \mid x_1 \in [0, 1] \wedge (\exists x_2 \in [0, 1 - x_1])((x_1, x_2) \in A)\},\\ &\sup\{x_2 \mid x_2 \in [0, 1] \wedge (\exists x_1 \in [0, 1 - x_2])((x_1, x_2) \in A\}).\end{aligned}$$

We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$.

From now on, we will assume that if $x \in L^*$, then x_1 and x_2 denote respectively the first and the second component of x, i.e. $x = (x_1, x_2)$.

Using this lattice, we easily see that with every intuitionistic fuzzy set $A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\}$ corresponds an L^*-fuzzy set, i.e. a mapping $A : U \to L^* : u \mapsto (\mu_A(u), \nu_A(u))$. In the sequel we will use the same notation for an intuitionistic fuzzy set and its associated L^*-fuzzy set. So for the intuitionistic fuzzy set A we will also use the notation $A(u) = (\mu_A(u), \nu_A(u))$.

Interpreting intuitionistic fuzzy sets as L^*-fuzzy sets gives way to greater flexibility in calculating with membership and non-membership degrees, since the pair formed by the two degrees is an element of L^*, and often allows to obtain more compact formulas. Moreover, some operators that are defined in the fuzzy case, such as fuzzy implicators, can be easily extended to the intuitionistic fuzzy case by using the lattice $(L^*, \leq_{L^*})$.

We also define the following set for further usage : $D = \{x \mid x \in L^* \wedge x_1 + x_2 = 1\}$, and the first and second projection mapping pr_1 and pr_2 on L^*, defined as $pr_1(x_1, x_2) = x_1$ and $pr_2(x_1, x_2) = x_2$, for all $(x_1, x_2) \in L^*$.

3 Intuitionistic fuzzy connectives

Using the lattice $(L^*, \leq_{L^*})$ the notions of triangular norm and conorm can be straightforwardly extended to the intuitionistic fuzzy case.

Definition 3.1 *[12, 11] An intuitionistic fuzzy triangular norm is a commutative, associative, increasing $(L^*)^2 - L^*$ mapping $\mathcal{T}$ satisfying $\mathcal{T}(x, 1_{L^*}) = x$, for all $x \in L^*$.*

Definition 3.2 *An intuitionistic fuzzy triangular conorm is a commutative, associative, increasing $(L^*)^2 - L^*$ mapping $\mathcal{S}$ satisfying $\mathcal{S}(x, 0_{L^*}) = x$, for all $x \in L^*$.*

Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm, then for any intuitionistic fuzzy negator $\mathcal{N}$, the mapping $\mathcal{T}^*$ defined by $\mathcal{T}^*(x,y) = \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y)))$, for all $x, y \in L^*$, is an intuitionistic fuzzy t-conorm. $\mathcal{T}^*$ is called the dual intuitionistic fuzzy t-conorm of $\mathcal{T}$ w.r.t. $\mathcal{N}$. Similarly, if $\mathcal{S}$ is an intuitionistic fuzzy t-conorm, then for any intuitionistic fuzzy negator $\mathcal{N}$, the mapping $\mathcal{S}^*$ defined by $\mathcal{S}^*(x,y) = \mathcal{N}(\mathcal{S}(\mathcal{N}(x), \mathcal{N}(y)))$, for all $x, y \in L^*$ is an intuitionistic fuzzy t-norm, called the dual intuitionistic fuzzy t-norm of $\mathcal{S}$ w.r.t. $\mathcal{N}$.

Intuitionistic fuzzy negators form an extension of fuzzy negators and are defined as follows.

Definition 3.3 *An intuitionistic fuzzy negator is any decreasing $L^* - L^*$ mapping $\mathcal{N}$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator.*

The mapping $\mathcal{N}_s$ defined by $\mathcal{N}_s(x_1, x_2) = (x_2, x_1)$, for all $(x_1, x_2) \in L^*$, will be called the standard negator.

In [12] Deschrijver, Cornelis and Kerre have established a representation theorem for involutive intuitionistic fuzzy negators : any involutive intuitionistic fuzzy negator can be represented using an involutive fuzzy negator, where a fuzzy negator is defined as a decreasing $[0,1] - [0,1]$ mapping satisfying $N(0) = 1$ and $N(1) = 0$.

Theorem 3.1 *[12] Let $\mathcal{N}$ be an involutive intuitionistic fuzzy negator, and let the $[0,1] - [0,1]$ mapping N be defined by, for $a \in [0,1]$, $N(a) = pr_1\mathcal{N}(a, 1-a)$. Then for all $x \in L^*$, $\mathcal{N}(x) = (N(1-x_2), 1 - N(x_1))$. Moreover, N is an involutive fuzzy negator. Conversely, if N is an involutive fuzzy negator, then the $L^* - L^*$ mapping $\mathcal{N}$ defined by, for all $x \in L^*$, $\mathcal{N}(x) = (N(1-x_2), 1 - N(x_1))$ is an involutive intuitionistic fuzzy negator.*

For instance, if $N = N_s$, where N_s denotes the fuzzy standard negator defined as, for all $x \in [0,1]$, $N_s(x) = 1 - x$, then we obtain the intuitionistic fuzzy standard negator $\mathcal{N}_s$.

In [12] we have also established a representation theorem for continuous increasing bijections on L^*.

Theorem 3.2 *[12] Let Φ be a continuous increasing bijection on L^*. Then there exists a continuous increasing bijection φ on $[0,1]$ such that, for all $x \in L^*$,*

$$\Phi(x) = (\varphi(x_1), 1 - \varphi(1 - x_2)). \tag{1}$$

Conversely, for any increasing bijection φ on $[0,1]$, the $L^* - L^*$ function Φ defined by (1) is an increasing bijection on L^*.

Using the lattice $(L^*, \leq_{L^*})$ it is also straightforward to extend the notions of fuzzy implicator and coimplicator.

Definition 3.4 [5] *An intuitionistic fuzzy implicator is an $(L^*)^2 - L^*$ mapping $\mathcal{I}$ which is decreasing in its first and increasing in its second component and which satisfies the border conditions $\mathcal{I}(0_{L^*}, 0_{L^*}) = \mathcal{I}(0_{L^*}, 1_{L^*}) = \mathcal{I}(1_{L^*}, 1_{L^*}) = 1_{L^*}$ and $\mathcal{I}(1_{L^*}, 0_{L^*}) = 0_{L^*}$.*

Definition 3.5 [12] *An intuitionistic fuzzy coimplicator is an $(L^*)^2 - L^*$ mapping $\mathcal{I}^c$ which is decreasing in its first and increasing in its second component and which satisfies the border conditions $\mathcal{I}^c(0_{L^*}, 0_{L^*}) = \mathcal{I}^c(1_{L^*}, 0_{L^*}) = \mathcal{I}^c(1_{L^*}, 1_{L^*}) = 0_{L^*}$ and $\mathcal{I}^c(0_{L^*}, 1_{L^*}) = 1_{L^*}$.*

In [5] Deschrijver and Cornelis have introduced the following classes of intuitionistic fuzzy implicators.

Definition 3.6 *Let $\mathcal{S}$ be an intuitionistic fuzzy t-norm and $\mathcal{N}$ an intuitionistic fuzzy negator. The S-implicator generated by $\mathcal{S}$ and $\mathcal{N}$ is the $(L^*)^2 - L^*$ mapping $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ defined by $\mathcal{I}_{\mathcal{S},\mathcal{N}}(x,y) = \mathcal{S}(\mathcal{N}(x), y)$, for all $x, y \in L^*$.*

Definition 3.7 *Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm. The R-implicator generated by $\mathcal{T}$ is the $(L^*)^2 - L^*$ mapping $\mathcal{I}_{\mathcal{T}}$ defined by $\mathcal{I}_{\mathcal{T}}(x,y) = \sup\{\gamma \mid \gamma \in L^* \wedge \mathcal{T}(x,\gamma) \leq_{L^*} y\}$, for all $x, y \in L^*$.*

4 *t*-representability

Intuitionistic fuzzy t-norms and t-conorms can be constructed using t-norms and t-conorms on $[0,1]$ in the following way. Let T be a t-norm and S a t-conorm. If $\mathcal{T}(a,b) \leq 1 - S(1 - a, 1 - b)$, for all $a, b \in [0,1]$, then the mapping $\mathcal{T}$ defined by $\mathcal{T}(x,y) = (T(x_1,y_1), S(x_2,y_2))$, for all $x, y \in L^*$, is an intuitionistic fuzzy t-norm, and the mapping $\mathcal{S}$ defined by $\mathcal{S}(x,y) = (S(x_1,y_1), T(x_2,y_2))$, for all $x, y \in L^*$, is an intuitionistic fuzzy t-conorm. Note that the condition $T(a,b) \leq 1 - S(1-a, 1-b)$, for all $a, b \in [0,1]$, is necessary and sufficient for $\mathcal{T}(x,y)$ and $\mathcal{S}(x,y)$ to be elements of L^* for all $x, y \in L^*$. We write $\mathcal{T} = (T, S)$ and $\mathcal{S} = (S, T)$.

Unfortunately the converse is not always true. It is not possible to find for any intuitionistic fuzzy t-norm $\mathcal{T}$ a t-norm T and a t-conorm S such that $\mathcal{T} = (T, S)$. Consider for instance the intuitionistic fuzzy t-norm $\mathcal{T}_W$ given by

$$\mathcal{T}_W(x,y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + 1 - y_1, y_2 + 1 - x_1)), \forall x, y \in L^*.$$

Let $x = (0.5, 0.3)$, $x' = (0.3, 0.3)$ and $y = (0.2, 0)$. Then $pr_2\mathcal{T}_W(x,y) = 0.5 \neq pr_2\mathcal{T}_W(x', y)$

= 0.7. Hence there exist no T and S such that $\mathcal{T} = (T, S)$, since this would imply that $pr_2 \mathcal{T}_W(x, y)$ is independent from x_1. In the sequel we will call this intuitionistic fuzzy t-norm the intuitionistic fuzzy Łukasiewicz t-norm.

To distinguish between these two kinds of intuitionistic fuzzy t-norms, we introduce the notion of t-representability [7]. We say that an intuitionistic fuzzy t-norm $\mathcal{T}$ is t-representable if there exist a t-norm T and a t-conorm S on $[0, 1]$ such that $\mathcal{T} = (T, S)$. An intuitionistic fuzzy t-conorm $\mathcal{S}$ is t-representable if there exist a t-norm T and a t-conorm S on $[0, 1]$ such that $\mathcal{S} = (S, T)$.

We have the following theorem assuring the t-representability of the dual of a given t-representable intuitionistic fuzzy t-norm or t-conorm.

Theorem 4.1 *[12] The dual intuitionistic fuzzy t-norm with respect to an involutive negator $\mathcal{N}$ on L^* of a t-representable intuitionistic fuzzy t-conorm is t-representable. The dual intuitionistic fuzzy t-conorm with respect to an involutive negator $\mathcal{N}$ on L^* of a t-representable intuitionistic fuzzy t-norm is t-representable.*

5 The residuation principle

We say that an intuitionistic fuzzy t-norm $\mathcal{T}$ satisfies the residuation principle if and only if, for all $x, y, z \in L^*$, $\mathcal{T}(x, z) \leq_{L^*} y \Leftrightarrow z \leq_{L^*} \mathcal{I}_{\mathcal{T}}(x, y)$, where $\mathcal{I}_{\mathcal{T}}$ denotes the residual implicator generated by $\mathcal{T}$.

Similarly we say that the intuitionistic fuzzy t-conorm $\mathcal{S}$ satisfies the residuation principle if and only if $\mathcal{S}(x, y) \geq_{L^*} z \Leftrightarrow y \geq_{L^*} \mathcal{I}^c_{\mathcal{S}}(x, z)$, where $\mathcal{I}^c_{\mathcal{S}}(x, z)$ is defined as $\mathcal{I}^c_{\mathcal{S}}(x, z) = \inf\{\gamma \mid \gamma \in L^* \wedge \mathcal{S}(x, \gamma) \geq_{L^*} z\}$.

Definition 5.1 *Let F be an arbitrary $L^* - L^*$ function and $a \in L^*$, then F is called intuitionistic fuzzy left-continuous in a iff*

$$(\forall \varepsilon > 0)(\exists \delta_1 > 0)(\exists \delta_2 > 0)(\forall x \in L^*)$$
$$(a_1 - \delta_1 < x_1 \leq a_1 \wedge a_2 \leq x_2 < a_2 + \delta_2 \Rightarrow d(F(x), F(a)) < \varepsilon). \tag{2}$$

F is called intuitionistic fuzzy right-continuous in a iff

$$(\forall \varepsilon > 0)(\exists \delta_1 > 0)(\exists \delta_2 > 0)(\forall x \in L^*)$$
$$(a_1 \leq x_1 < a_1 + \delta_1 \wedge a_2 - \delta_2 < x_2 \leq a_2 \Rightarrow d(F(x), F(a)) < \varepsilon). \tag{3}$$

Theorem 5.1 *Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm. If $\sup\limits_{z \in Z} \mathcal{T}(x, z) = \mathcal{T}\left(x, \sup\limits_{z \in Z} z\right)$, for all non-empty subsets Z of L^*, then $\mathcal{T}$ is intuitionistic fuzzy left-continuous.*

Let $\mathcal{S}$ be an intuitionistic fuzzy t-conorm. If $\inf\limits_{z \in Z} \mathcal{S}(x, z) = \mathcal{S}\left(x, \inf\limits_{z \in Z} z\right)$, for all non-empty subsets Z of L^*, then $\mathcal{S}$ is intuitionistic fuzzy right-continuous.

Theorem 5.2 *Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm. Then $\mathcal{T}$ satisfies the residuation principle if and only if $\sup_{z\in Z}\mathcal{T}(x,z) = \mathcal{T}\Big(x, \sup_{z\in Z} z\Big)$, for any $x \in L^*$ and any subset Z of L^*.*

Let $\mathcal{S}$ be an intuitionistic fuzzy t-conorm. Then $\mathcal{S}$ satisfies the residuation principle if and only if $\inf_{z\in Z}\mathcal{S}(x,z) = \mathcal{S}\Big(x, \inf_{z\in Z} z\Big)$, for any $x \in L^*$ and any subset Z of L^*.

Theorem 5.3 *Let $\mathcal{T}$ be a t-representable intuitionistic fuzzy t-norm. Then $\mathcal{T}$ is intuitionistic fuzzy left-continuous if and only if $\mathcal{T}$ satisfies the residuation principle.*

Let $\mathcal{S}$ be a t-representable intuitionistic fuzzy t-conorm. Then $\mathcal{S}$ is intuitionistic fuzzy right-continuous if and only if $\mathcal{S}$ satisfies the residuation principle.

In general from intuitionistic fuzzy left-continuity it cannot be deduced that an intuitionistic fuzzy t-norm satisfies the residuation principle. Consider for instance the intuitionistic fuzzy t-norm $\mathcal{T}$ defined as $\mathcal{T}(x,y) = (\max(0, x_1 + y_1 - x_2y_2 - 1), \min(1, x_2 + y_2))$, for all $x, y \in L^*$.

We obtain that

$$\begin{aligned}\mathcal{I}_\mathcal{T}(x,z) &= \sup\{y \mid y \in L^* \wedge \mathcal{T}(x,y) \leq_{L^*} z\}\\ &= \Big(\min\Big(1, 1 + x_2 - z_2, \max\Big(1 + z_1 - x_1 + x_2 \max(0, z_2 - x_2), 1 + \frac{z_1 - x_1}{1 + x_2}\Big)\Big),\\ &\qquad \max(0, z_2 - x_2)\Big).\end{aligned}$$

Let now $x = (0.5, 0.4)$ and $y = (0.3, 0.5)$. Then $z = \mathcal{I}_\mathcal{T}(x,y) = (\frac{6}{7}, 0.1) = (0.857142\ldots, 0.1)$, but $\mathcal{T}(x,z) = (0.317142\ldots, 0.5) \not\leq_{L^*} y$. Hence $\mathcal{T}$ is a continuous intuitionistic fuzzy t-norm which does not satisfy the residuation principle.

6 A representation of intuitionistic fuzzy t-norms

An intuitionistic fuzzy t-norm $\mathcal{T}$ is called Archimedean if and only if, for all $x \in L^* \backslash \{0_{L^*}, 1_{L^*}\}$, $\mathcal{T}(x,x) <_{L^*} x$.

An intuitionistic fuzzy t-norm $\mathcal{T}$ is called nilpotent if and only if there exist $x, y \in L^* \backslash \{0_{L^*}\}$ such that $\mathcal{T}(x,y) = 0_{L^*}$. $\mathcal{T}$ is called intuitionistic fuzzy nilpotent if and only if there exist $x, y \in L^*$ such that x_1 and y_1 are non-zero and such that $\mathcal{T}(x,y) = 0_{L^*}$.

Theorem 6.1 *Let $\mathcal{T}$ be an $(L^*)^2 - L^*$ mapping. Then the following are equivalent :*

(i) $\mathcal{T}$ is a continuous, Archimedean, intuitionistic fuzzy nilpotent intuitionistic fuzzy t-norm satisfying the residuation principle, $\mathcal{I}_\mathcal{T}(D, D) \subseteq D$ and $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$;

(ii) there exists a continuous increasing bijection φ on $[0,1]$ such that, for all $x, y \in L^*$,

$$\begin{aligned}\mathcal{T}(x,y) &= (\varphi^{-1}\max(0, \varphi(x_1) + \varphi(y_1) - 1),\\ &\quad 1 - \varphi^{-1}\max(0, \varphi(x_1) + \varphi(1 - y_2) - 1, \varphi(y_1) + \varphi(1 - x_2) - 1));\end{aligned} \tag{4}$$

(iii) there exists a continuous increasing bijection Φ on L^* such that $\mathcal{T} = \Phi^{-1} \circ \mathcal{T}_W \circ (\Phi, \Phi)$.

Note that if $\mathcal{T}$ satisfies (ii), then $\mathcal{I}_{\mathcal{T}}(x,z) = (\varphi^{-1}\min(1, \varphi(z_1) + 1 - \varphi(x_1), 1 - \varphi(1 - x_2) + \varphi(1 - z_2)), 1 - \varphi^{-1}(1 - \max(0, \varphi(x_1) - \varphi(1 - z_2))))$. Moreover $\mathcal{N}(x) = \mathcal{I}_{\mathcal{T}}(x, 0_{L^*}) = (\varphi^{-1}(1-\varphi(1-x_2)), 1-\varphi^{-1}(1-\varphi(x_1)))$, so $\mathcal{N}(x) = (N(1-x_2), 1-N(x_1))$, with $N = \varphi^{-1} \circ N_s \circ \varphi$. It also follows that $\mathcal{T}(D, D) \subseteq D$.

7 A representation of intuitionistic fuzzy t-conorms

An intuitionistic fuzzy t-conorm $\mathcal{S}$ is called Archimedean if and only if, for all $x \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, $\mathcal{S}(x,x) >_{L^*} x$.

An intuitionistic fuzzy t-conorm $\mathcal{S}$ is called nilpotent if and only if there exist $x, y \in L^* \setminus \{1_{L^*}\}$ such that $\mathcal{S}(x,y) = 1_{L^*}$. $\mathcal{S}$ is called intuitionistic fuzzy nilpotent if and only if there exist $x, y \in L^*$ such that x_2 and y_2 are non-zero and such that $\mathcal{S}(x,y) = 1_{L^*}$.

Define the $(L^*)^2 - L^*$ mapping $\mathcal{S}_W$ as $\mathcal{S}_W(x,y) = (\min(1, 1 - x_2 + y_1, 1 - y_2 + x_1), 1 - \min(1, 1 - x_2 + 1 - y_2))$, for all $x, y \in L^*$. Then $\mathcal{S}_W$ is the dual intuitionistic fuzzy t-conorm of $\mathcal{T}_W$ w.r.t. the standard negator $\mathcal{N}_s$. The residual coimplicator $\mathcal{I}^c_{\mathcal{S}_W}$ of $\mathcal{S}_W$ is given by $\mathcal{I}^c_{\mathcal{S}_W}(x,z) = (\max(0, z_1 + x_2 - 1), \min(1, z_2 + 1 - x_2, x_1 + 1 - z_1))$. It holds that $\mathcal{I}^c_{\mathcal{S}_W} = \mathcal{N}_s \circ \mathcal{I}_{\mathcal{T}_W} \circ (\mathcal{N}_s, \mathcal{N}_s)$.

Theorem 7.1 *Let $\mathcal{S}$ be an $(L^*)^2 - L^*$ mapping. Then the following are equivalent :*

(i) $\mathcal{S}$ is a continuous, Archimedean, intuitionistic fuzzy nilpotent intuitionistic fuzzy t-conorm satisfying the residuation principle, $\mathcal{I}^c_{\mathcal{S}}(D, D) \subseteq D$ and $\mathcal{S}((0,0),(0,0)) = 1_{L^*}$;

(ii) there exists a continuous increasing bijection φ on $[0,1]$ such that, for all $x, y \in L^*$,

$$\begin{aligned}\mathcal{S}(x,y) &= (\varphi^{-1}\min(1, \varphi(1-x_2) + \varphi(y_1), \varphi(1-y_2) + \varphi(x_1)),\\ &\quad 1 - \varphi^{-1}\min(1, \varphi(1-x_2) + \varphi(1-y_2)));\end{aligned} \tag{5}$$

(iii) there exists a continuous increasing bijection Φ on L^* such that $\mathcal{S} = \Phi^{-1} \circ \mathcal{S}_W \circ (\Phi, \Phi)$.

For any intuitionistic fuzzy coimplicator $\mathcal{I}^c$, we call the $L^* - L^*$ mapping $\mathcal{N}$ defined by $\mathcal{N}(x) = \mathcal{I}^c(x, 1_{L^*})$ the negator induced by $\mathcal{I}^c$. This is indeed an intuitionistic fuzzy negator since $\mathcal{I}^c$ is decreasing in its first component, $\mathcal{I}^c(0_{L^*}, 1_{L^*}) = 1_{L^*}$ and $\mathcal{I}^c(1_{L^*}, 1_{L^*}) = 0_{L^*}$.

Note that if $\mathcal{S}$ satisfies (ii) in Theorem 7.1, then $\mathcal{I}^c_{\mathcal{S}}(x,z) = (\varphi^{-1}\max(0, \varphi(z_1) - \varphi(1 - x_2)), 1 - \varphi^{-1}(1 - \min(1, 1 - \varphi(1 - z_2) + \varphi(1 - x_2), \varphi(x_1) + 1 - \varphi(z_1))))$. Moreover $\mathcal{N}(x) = \mathcal{I}^c_{\mathcal{S}}(x, 1_{L^*}) = (\varphi^{-1}(1 - \varphi(1 - x_2)), 1 - \varphi^{-1}(1 - \varphi(x_1)))$. So $\mathcal{N}(x) = (N(1 - x_2), 1 - N(x_1))$, with $N = \varphi^{-1} \circ N_s \circ \varphi$.

Theorem 7.2 *Let $\mathcal{S}$ be an intuitionistic fuzzy t-conorm and assume there exists a continuous increasing bijection φ on $[0,1]$ such that (5) holds. Let $\mathcal{N}$ be the negator induced by $\mathcal{I}^c_{\mathcal{S}}$. Then the dual intuitionistic fuzzy t-norm $\mathcal{T}$ of $\mathcal{S}$ w.r.t. $\mathcal{N}$ satisfies (4) for the same φ.*

Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm and assume there exists a continuous increasing bijection φ on $[0,1]$ such that (4) holds. Let $\mathcal{N}$ be the negator induced by $\mathcal{I}_\mathcal{T}$. Then the dual intuitionistic fuzzy t-conorm $\mathcal{S}$ of $\mathcal{T}$ w.r.t. $\mathcal{N}$ satisfies (5) for the same φ.

8 A representation of intuitionistic fuzzy implicators

The suitability of IF implicators for a variety of purposes can be assessed using the (generalized) criteria introduced by Smets and Magrez in [19] :

Definition 8.1 (Axioms of Smets and Magrez for an IF implicator $\mathcal{I}$)

(A.1) $(\forall y \in L^*)(\mathcal{I}(.,y)$ *is decreasing*$)$
$(\forall x \in L^*)(\mathcal{I}(x,.)$ *is increasing*$)$ (monotonicity laws)

(A.2) $(\forall x \in L^*)(\mathcal{I}(1_{L^*}, x) = x)$ (neutrality principle)

(A.3) $(\forall (x,y) \in (L^*)^2)(\mathcal{I}(x,y) = \mathcal{I}(\mathcal{N}(y), \mathcal{N}(x)))$ (contrapositivity w.r.t. an IF negator $\mathcal{N}$)

(A.4) $(\forall (x,y,z) \in (L^*)^3)(\mathcal{I}(x, \mathcal{I}(y,z)) = \mathcal{I}(y, \mathcal{I}(x,z)))$ (interchangeability principle)

(A.5) $(\forall (x,y) \in (L^*)^2)(x \leq_{L^*} y \Leftrightarrow \mathcal{I}(x,y) = 1_{L^*})$ (confinement principle)

(A.6) $\mathcal{I}$ *is continuous* (continuity)

Clearly (A.1) is already contained in the definition of intuitionistic fuzzy implicator. For any implicator $\mathcal{I}$, the mapping $\mathcal{N}$ defined as $\mathcal{N}(x) = \mathcal{I}(x, 0_{L^*})$, for all $x \in L^*$, is called the negator induced by $\mathcal{I}$. In [11] we have proven that if an intuitionistic fuzzy implicator $\mathcal{I}$ satisfies (A.2), (A.3), (A.4), then $\mathcal{I}$ is an S-implicator. In [7] we have also shown that if an implicator $\mathcal{I}$ is contrapositive w.r.t. an intuitionistic fuzzy negator $\mathcal{N}$, then necessarily $\mathcal{N}$ is the negator induced by $\mathcal{I}$ and $\mathcal{N}$ is involutive. We have also shown that if $\mathcal{S}$ is t-representable and $\mathcal{N}$ is involutive, then $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ does not satisfy (A.5). From all these considerations follows that a t-representable implicator cannot satisfy all the Smets-Magrez axioms at once.

If an intuitionistic fuzzy implicator satisfies all the Smets-Magrez axioms at once, then we call it an intuitionistic fuzzy Łukasiewicz implicator. Hence only a non t-representable intuitionistic fuzzy implicator can be an intuitionistic fuzzy Łukasiewicz implicator. An example of such an implicator is the R-implicator generated by $\mathcal{T}_W$, which is given by, for all $x, y \in L^*$,

$$\mathcal{I}_{\mathcal{T}_W}(x,y) = (\min(1, z_1 + 1 - x_1, x_2 + 1 - z_2), \max(0, z_2 + x_1 - 1)).$$

Now the question arises whether we can find all the intuitionistic fuzzy Łukasiewicz implicators. In [8] we have found the following result.

Theorem 8.1 *Let $\mathcal{I}$ be an intuitionistic fuzzy implicator such that $\mathcal{I}(D,D) \subseteq D$. Then the following are equivalent :*

(i) $\mathcal{I}$ is an intuitionistic fuzzy Łukasiewicz implicator;

(ii) there exists a continuous increasing bijection φ on $[0,1]$ such that, for all $x, y \in L^*$,

$$\begin{aligned}\mathcal{I}_{\mathcal{T}}(x,z) &= (\varphi^{-1}\min(1, \varphi(z_1) + 1 - \varphi(x_1), 1 - \varphi(1-x_2) + \varphi(1-z_2)),\\ &\quad 1 - \varphi^{-1}(1 - \max(0, \varphi(x_1) - \varphi(1-z_2))));\end{aligned}$$

(iii) there exists a continuous increasing bijection Φ on L^* such that $\mathcal{I} = \Phi^{-1} \circ \mathcal{I}_{\mathcal{T}_W} \circ (\Phi, \Phi)$.

9 Conclusion

In fuzzy set theory a t-norm satisfies the residuation principle if and only if it is left-continuous. We have shown that for intuitionistic fuzzy t-norms intuitionistic fuzzy left-continuity is a necessary but not sufficient condition for the residuation principle to hold. We have introduced the intuitionistic fuzzy Łukasiewicz t-norm $\mathcal{T}_W$ and established the necessary and sufficient conditions for an intuitionistic fuzzy t-norm $\mathcal{T}$ such that there exists a bijection Φ on L^* such that $\mathcal{T}$ is the Φ-transform of $\mathcal{T}_W$. Similarly we have found a representation theorem for a subclass of intuitionistic fuzzy t-conorms. In order to prove these representation theorems we had to establish a representation for involutive negators and continuous increasing bijections. Finally we have established a representation theorem for intuitionistic fuzzy Łukasiewicz implicators.

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