Triangle and square: a comparison

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Abstract

In this paper we study some of the characteristics of and differences between the evaluation structures of intuitionistic fuzzy set theory ("triangle") and fuzzy four-valued or Belnap logic ("square").

Keywords: intuitionistic fuzzy set, fuzzy four-valued logic, L-fuzzy set, lattice L^*, lattice $L_\square$.

1 Introduction

Intuitionistic fuzzy set theory [1, 2] is an extension of Zadeh's fuzzy set theory in which to any element u in the universe U not only a membership degree μ but also a non-membership degree ν is assigned. In fuzzy set theory the non-membership degree is assumed to be equal to one minus the membership degree, in intuitionistic fuzzy set theory only the weaker constraint $\nu \leq 1 - \mu$ is enforced. The amount of indeterminacy, or "missing information", is modelled by the number $\pi = 1 - \mu - \nu$.

Just like the relationship between classical logic and set theory was exploited in fuzzy set theory to define "fuzzy logics" (in a narrow sense), so we may also introduce a notion of "intuitionistic fuzzy logics": with a proposition p a degree of truth $\mu(p)$ and a degree of falsity $\nu(p)$ is associated, such that $\mu(p) + \nu(p) \leq 1$. This idea is elaborated in e.g. [3].

It should be mentioned that the term "intuitionistic" is to be read in a "broad" sense here, alluding loosely to the denial of the law of the excluded middle on element level (since $\mu(p) + \nu(p) < 1$ is possible). A "narrow", graded extension of intuitionistic logic proper has also been proposed and is due to Takeuti and Titani [13]. It bears no relationship to the notion of intuitionistic fuzzy logic described in this paper.

Intuitionistic fuzzy logic can be generalized by dropping the restriction $\mu(p) + \nu(p) \leq 1$, and instead draw $(\mu(p), \nu(p))$ from $[0, 1]^2$. This extension is called fuzzy four-valued logic or fuzzy Belnap logic, as it extends the logical evaluation structure $\mathcal{FOUR}$ introduced by Belnap [4] and shown in Figure 1.

Figure 1: Belnap's logical evaluation structure $\mathcal{FOUR}$.

In $\mathcal{FOUR}$ we have four epistemic states `true` (T), `false` (F), `unknown` (U) and `contradiction` (C) that can represent an agent's beliefs with respect to the truth of a proposition. By mapping the epistemic states on the angular points of the unit square as follows: $T \to (1,0)$, $F \to (0,1)$, $U \to (0,0)$ and $C \to (1,1)$, and by drawing values from the entire unit square, we obtain fuzzy four-valued logic. Since also in intuitionistic fuzzy logic, `true` corresponds to $(1,0)$, `false` to $(0,1)$ and `unknown` to $(0,0)$, and since by the restriction of truth and falsity degrees the state `contradiction` is not allowed, it is clear that its evaluation structure is a triangle that takes up only (the consistent) half of the unit square. This is also discussed in [5].

In this paper, we compare the evaluation structures of intuitionistic fuzzy logic and fuzzy four-valued logic from a mathematical point of view. The respective evaluation structures "triangle" and "square" will be viewed as particular complete lattices. This allows us to define graded versions of the logical connectives (negators, t-norms, t-conorms and implicators generalizing classical negation, conjunction, disjunction and implication respectively). We show that both structures are mathematically different and we will emphasize this observation by showing that the graded connectives exhibit different properties in both structures.

2 The lattice L^*

Consider the set L^* (see Figure 2) and order relation $\leq_{L^*}$ defined by:

$$L^* = \{(x_1, x_2) \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2, \quad \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ is a complete lattice [10]. We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$.

Note that if, for $x = (x_1, x_2), y = (y_1, y_2) \in L^*$, ($x_1 < y_1$ and $x_2 < y_2$) or ($x_1 > y_1$ and $x_2 > y_2$), then x and y are incomparable w.r.t $\leq_{L^*}$, denoted as $x \|_{L^*} y$.

From now on, we will assume that if $x \in L^*$, then x_1 and x_2 denote respectively the first and the second component of x, i.e. $x = (x_1, x_2)$.

Figure 2: The shaded area constitutes the set L^*.

We also define the following set for further use: $D = \{x \mid x \in L^* \text{ and } x_1 + x_2 = 1\}$, and the first and second projection mapping pr_1 and pr_2 on L^*, defined as $\mathrm{pr}_1(x_1, x_2) = x_1$ and $\mathrm{pr}_2(x_1, x_2) = x_2$, for all $(x_1, x_2) \in L^*$.

Define the set $L_\Box$ (see Figure 3) and order relation $\leq_\Box$ as

$$L_\Box = [0,1]^2,$$

$$(x_1, x_2) \leq_\Box (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2, \quad \forall (x_1, x_2), (y_1, y_2) \in L_\Box.$$

Figure 3: The shaded area constitutes the set $L_\Box$.

Then $(L_\Box, \leq_\Box)$ is a complete lattice. Note that its units are equal to the units of L^*: $0_\Box = 0_{L^*}$ and $1_\Box = 1_{L^*}$.

Similarly as in L^*, if for $x = (x_1, x_2), y = (y_1, y_2) \in L^*$ it holds that ($x_1 < y_1$ and $x_2 < y_2$) or ($x_1 > y_1$ and $x_2 > y_2$), then x and y are incomparable w.r.t $\leq_\Box$, denoted as $x \|_\Box y$.

Theorem 2.1 *There does not exist an order-isomorphism between the lattices $(L^*, \leq_{L^*})$ and $(L_\square, \leq_\square)$, i.e. there does not exist a bijection Φ from L^* to $L_\square$ such that $x \leq_{L^*} y \Leftrightarrow \Phi(x) \leq_\square \Phi(y)$, for all $x, y \in L^*$.*

In spite of the fact that $L^* \subseteq L_\square$, the ordering is the same and the units are the same, from Theorem 2.1 it follows that $(L^*, \leq_{L^*})$ and $(L_\square, \leq_\square)$ have a different structure. In the rest of the paper we will illustrate this difference by contrasting operations on L^* and $L_\square$ and their respective properties.

3 Negators

Definition 3.1 *A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \to L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator.*

Note that the mapping $\mathcal{N}_s$ defined by $\mathcal{N}_s(x_1, x_2) = (x_2, x_1)$, for all $(x_1, x_2) \in L^*$, is an involutive negator on L^*. We will call $\mathcal{N}_s$ the standard negator on L^*. The following theorem was established in [9].

Theorem 3.1 [9] *Let $\mathcal{N}$ be a negator on L^* and let the $[0,1] \to [0,1]$ mapping N be defined by $N(a) = \mathrm{pr}_1 \mathcal{N}(a, 1-a)$, for all $a \in [0, 1$. Then $\mathcal{N}$ is involutive if and only if N is involutive and for all $x \in L^*$:*

$$\mathcal{N}(x) = (N(1 - x_2), 1 - N(x_1)).$$

Definition 3.2 *A negator on $L_\square$ is any decreasing mapping $\mathfrak{N} : L_\square \to L_\square$ satisfying $\mathfrak{N}(0_\square) = 1_\square$ and $\mathfrak{N}(1_\square) = 0_\square$. If $\mathfrak{N}(\mathfrak{N}(x)) = x$, for all $x \in L_\square$, then $\mathfrak{N}$ is called an involutive negator.*

Lemma 3.1 *For any involutive negator $\mathfrak{N}$ on $L_\square$ one of the following holds:*

(i) $\mathfrak{N}(0,0) = (0,0)$ *and* $\mathfrak{N}(1,1) = (1,1)$; *or*

(ii) $\mathfrak{N}(0,0) = (1,1)$ *and* $\mathfrak{N}(1,1) = (0,0)$.

Theorem 3.2 *Let $\mathfrak{N}$ be a negator on $L_\square$.*

(i) If $\mathfrak{N}(0,0) = (0,0)$, then let φ be the $[0,1] \to [0,1]$ mapping defined as $\varphi(a) = \mathrm{pr}_1 \mathfrak{N}(0, a)$, for all $a \in [0,1]$. Then $\mathfrak{N}$ is involutive if and only if φ is an increasing permutation of $[0,1]$ and, for all $x \in L_\square$,

$$\mathfrak{N}(x) = (\varphi(x_2), \varphi^{-1}(x_1)).$$

(ii) If $\mathfrak{N}(0,0) = (1,1)$, then let N_1 and N_2 be the $[0,1] \to [0,1]$ mappings defined as $N_1(a) = \mathrm{pr}_1 \mathfrak{N}(a, 0)$ and $N_2(a) = \mathrm{pr}_2 \mathfrak{N}(0, a)$, for all $a \in [0,1]$. Then $\mathfrak{N}$ is involutive if and only if N_1 and N_2 are involutive negators on $[0,1]$ and, for all $x \in L_\square$,

$$\mathfrak{N}(x) = (N_1(x_1), N_2(x_2)).$$

If in the first case $\varphi(x_1) = x_1$, for all $x_1 \in [0,1]$, then we obtain $\mathfrak{N}(x) = (x_2, x_1)$, i.e. the straightforward extension of the standard negator $\mathcal{N}_s$ on L^* to $L_\square$. We denote this negator by $\mathfrak{N}_s^1$. If in the second case $N_1(x_1) = N_2(x_1) = 1 - x_1$, for all $x_1 \in [0,1]$, then we obtain $\mathfrak{N}(x) = (1 - x_1, 1 - x_2)$. We denote this negator by $\mathfrak{N}_s^2$.

Let in Theorem 3.2 $N(a) = \varphi(1 - a)$, for all $a \in [0,1]$. Then N is a bijective negator on $[0,1]$ and $\mathfrak{N}(x) = (N(1 - x_2), 1 - N^{-1}(x_1))$, for all $x \in L_\square$. While for negators on L^* the corresponding negator N is involutive, this is not necessarily the case for negators on $L_\square$. Note also that the case (ii) in Theorem 3.2 cannot occur in L^*.

It is noteworthy that in [11] the mapping $\mathfrak{N}_s^1$ is called strong negation and the mapping $\mathfrak{N}_s^2$ is referred to as a complementation.

We show another difference between $(L^*, \leq_{L^*})$ and $(L_\square, \leq_\square)$ by investigating the presence of a Kleene negator in both lattices.

Definition 3.3 *Let $(L, \vee, \wedge, 0, 1)$ be a distributive bounded lattice. An mapping $N : L \to L$ is called a Kleene negator if and only if, for all $a, b \in L$:*

(K1) $a = N(N(a))$,

(K2) $N(a \vee b) = N(a) \wedge N(b)$,

(K3) $a \wedge N(a) \leq b \vee N(b)$.

Assume there exists a Kleene negator $\mathcal{N}$ on L^*, then by (K1) $\mathcal{N}$ is involutive. From Theorem 3.1 it follows that there exists an involutive negator N on $[0,1]$ such that, for all $x \in L^*$,

$$\mathcal{N}(x) = (N(1-x_2), 1-N(x_1)).$$

Hence (K3) is equivalent to, for all $a, b \in L^*$,

$$(\min(a_1, N(1-a_2)), \max(a_2, 1-N(a_1))) \leq_{L^*} (\max(b_1, N(1-b_2)), \min(b_2, 1-N(b_1))).$$

Let $a = (0.5, 0.5)$ and $b = (0,0)$, then we obtain $(\min(0.5, N(0.5)), \max(0.5, 1-N(0.5))) \leq_{L^*} (0,0)$, which implies $N(0.5) = 0$. From the involutivity of N it follows $0.5 = N(0) = 1$, which is a contradiction. Hence there does not exist a Kleene negator on L^*.

We now consider the negator $\mathfrak{N}_s^2$ on $L_\square$, then (K3) is equivalent to, for all $a, b \in L_\square$,

$$(\min(a_1, 1-a_1), \max(a_2, 1-a_2)) \leq_\square (\max(b_1, 1-b_1), \min(b_2, 1-b_2)).$$

Since $\min(a_1, 1-a_1) \leq \frac{1}{2} \leq \max(b_1, 1-b_1)$ and similarly for the second component, it follows that $\mathfrak{N}_s^2$ is a Kleene negator on $L_\square$.

4 Triangular norms and conorms

Definition 4.1 [8, 9] *A triangular norm (t-norm) on L^* is a commutative, associative, increasing mapping $\mathcal{T} : (L^*)^2 \to L^*$ such that $\mathcal{T}(1_{L^*}, x) = x$, for all $x \in L^*$.*

A triangular conorm (t-conorm) on L^* is a commutative, associative, increasing mapping $\mathcal{S} : (L^*)^2 \to L^*$ such that $\mathcal{S}(0_{L^*}, x) = x$, for all $x \in L^*$.

Some examples of t-norms and t-conorms on L^* are, for $x, y \in L^*$:

(i) $\inf(x,y) = (\min(x_1,y_1), \max(x_2,y_2))$,

(ii) $\mathcal{T}_w(x,y) = (\max(0, x_1+y_1-1), \min(1, x_2+y_2))$,

(iii) $\mathcal{S}_w(x,y) = (\min(1, x_1+y_1), \max(0, x_2+y_2-1))$,

(iv) $\mathcal{T}_W(x,y) = (\max(0, x_1+y_1-1), \min(1, x_2+1-y_1, y_2+1-x_1))$,

(v) $\mathcal{S}_W(x,y) = (\min(1, x_1+1-y_2, y_1+1-x_2), \max(0, x_2+y_2-1))$.

Note that both $\mathcal{T}_w$ and $\mathcal{T}_W$ are extensions to L^* of the Łukasiewicz t-norm T_W on $[0,1]$ defined by $T_W(x,y) = \max(0, x+y-1)$, for all $x, y \in [0,1]$.

Definition 4.2 *A triangular norm (t-norm) on $L_\square$ is a commutative, associative, increasing mapping $\mathfrak{T} : (L_\square)^2 \to L_\square$ such that $\mathfrak{T}(1_\square, x) = x$, for all $x \in L_\square$.*

A triangular conorm (t-conorm) on $L_\square$ is a commutative, associative, increasing mapping $\mathfrak{S} : (L_\square)^2 \to L_\square$ such that $\mathfrak{S}(0_\square, x) = x$, for all $x \in L_\square$.

Some examples of t-norms on $L_\square$ are, for $x, y \in L_\square$:

(i) $\inf(x,y) = (\min(x_1,y_1), \max(x_2,y_2))$,

(ii) $\mathfrak{T}_W(x,y) = (\max(0, x_1+y_1-1), \min(1, x_2+y_2))$,

(iii) $\mathfrak{S}_W(x,y) = (\min(1, x_1+y_1), \max(0, x_2+y_2-1))$.

Note that the mapping $\mathfrak{T}$ defined by $\mathfrak{T}(x,y) = (\max(0, x_1+y_1-1), \min(1, x_2+1-y_1, y_2+1-x_1))$, for all $x, y \in L_\square$, is not a t-norm on $L_\square$ since $\mathfrak{T}(1_\square, y) = (y_1, \min(y_2, 1-y_1)) \neq y$ as soon as $y_2 > 1-y_1$. Idem for the analogon of $\mathcal{S}_W$.

Definition 4.3 [8, 9] *A t-norm $\mathcal{T}$ (resp. a t-conorm $\mathcal{S}$) on L^* is called t-representable iff there exist a t-norm T and a t-conorm S on $[0,1]$ such that, for all $x, y \in L^*$,*

$$\mathcal{T}(x,y) = (T(x_1,y_1), S(x_2,y_2)),$$
$$(\textit{resp. } \mathcal{S}(x,y) = (S(x_1,y_1), T(x_2,y_2))).$$

We note $\mathcal{T} = (T, S)$ (resp. $\mathcal{S} = (S, T)$).

Similarly, t-representability can be defined for t-norms and t-conorms on $L_\square$. Examples of t-representable t-norms and t-conorms are inf, $\mathcal{T}_w$, $\mathcal{S}_w$, $\mathfrak{T}_W$ and $\mathfrak{S}_W$. Note that $\mathcal{T}_W$ and $\mathcal{S}_W$ are not t-representable.

Definition 4.4 *The dual of a t-norm $\mathcal{T}$ (resp. t-conorm $\mathcal{S}$) on L^* w.r.t. a negator $\mathcal{N}$ on L^* is the mapping $\mathcal{T}^*$ (resp. $\mathcal{S}^*$) defined by, for $x, y \in L^*$,*

$$\mathcal{T}^*(x,y) = \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y))),$$
$$(\textit{resp. } \mathcal{S}^*(x,y) = \mathcal{N}(\mathcal{S}(\mathcal{N}(x), \mathcal{N}(y)))).$$

The dual of a t-norm or t-conorm on $L_\square$ is defined in a similar way.

For example, $\mathcal{S}_w$ is the dual of $\mathcal{T}_w$ w.r.t. $\mathcal{N}_s$, $\mathcal{S}_W$ is the dual of $\mathcal{T}_W$ w.r.t. $\mathcal{N}_s$, and $\mathfrak{S}_W$ is the dual of $\mathfrak{T}_W$ w.r.t. both $\mathfrak{N}_s^1$ and $\mathfrak{N}_s^2$.

5 Implicators

Definition 5.1 [6, 8] *An implicator on L^* is a mapping $\mathcal{I} : (L^*)^2 \to L^*$ such that*

$$\mathcal{I}(0_{L^*}, 0_{L^*}) = 1_{L^*}, \quad \mathcal{I}(0_{L^*}, 1_{L^*}) = 1_{L^*},$$
$$\mathcal{I}(1_{L^*}, 1_{L^*}) = 1_{L^*}, \quad \mathcal{I}(1_{L^*}, 0_{L^*}) = 0_{L^*},$$

and such that $\mathcal{I}(., x)$ is decreasing and $\mathcal{I}(x, .)$ increasing, for all $x \in L^*$.

There are two important subclasses of implicators on L^*.

Definition 5.2 [6, 8] *Let $\mathcal{S}$ be a t-conorm on L^* and $\mathcal{N}$ a negator on L^*. The S-implicator generated by $\mathcal{S}$ and $\mathcal{N}$ is the mapping $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ defined as, for $x, y \in L^*$:*

$$\mathcal{I}_{\mathcal{S},\mathcal{N}}(x,y) = \mathcal{S}(\mathcal{N}(x), y).$$

Definition 5.3 [6, 8] *Let $\mathcal{T}$ be a t-norm on L^*. The residual implicator (R-implicator) generated by $\mathcal{T}$ is the mapping $\mathcal{I}_\mathcal{T}$ defined as, for $x, y \in L^*$:*

$$\mathcal{I}_\mathcal{T}(x,y)$$
$$= \sup\{\gamma \mid \gamma \in L^* \textit{ and } \mathcal{T}(x,\gamma) \leq_{L^*} y\}.$$

Some examples of S- and R-implicators on L^* are, for $x, y \in L^*$:

(i) $\mathcal{I}_{\inf}(x,y)$
$$= \begin{cases} 1_{L^*}, & \text{if } x_1 \leq y_1 \text{ and } x_2 \geq y_2, \\ (1-y_2, y_2), & \text{if } x_1 \leq y_1 \text{ and } x_2 < y_2, \\ (y_1, 0), & \text{if } x_1 > y_1 \text{ and } x_2 \geq y_2, \\ (y_1, y_2), & \text{if } x_1 > y_1 \text{ and } x_2 < y_2; \end{cases}$$

(ii) $\mathcal{I}_{\mathcal{T}_w}(x,y) = (\min(1, y_1+1-x_1, x_2+1-y_2), \max(0, y_2 - x_2))$;

(iii) $\mathcal{I}_{\mathcal{S}_w,\mathcal{N}_s}(x,y) = (\min(1, x_2 + y_1), \max(0, x_1 + y_2 - 1))$;

(iv) $\mathcal{I}_{\mathcal{T}_W}(x,y) = \mathcal{I}_{\mathcal{S}_W,\mathcal{N}_s}(x,y) = (\min(1, y_1 + 1 - x_1, x_2 + 1 - y_2), \max(0, y_2 + x_1 - 1))$.

From fuzzy set theory we know that the S-implicator generated by the dual of T_W and N_s (where N_s is defined as $N_s(x) = 1 - x$, for all $x \in [0,1]$) is equal to the R-implicator of T_W. From the above it follows that a similar result holds for the non-t-representable extension $\mathcal{T}_W$ of T_W to L^*, but not for its t-representable extension $\mathcal{T}_w$.

Definition 5.4 *An implicator on $L_\square$ is a mapping $\mathfrak{I} : (L_\square)^2 \to L_\square$ such that*

$$\mathfrak{I}(0_\square, 0_\square) = 1_\square, \quad \mathfrak{I}(0_\square, 1_\square) = 1_\square,$$
$$\mathfrak{I}(1_\square, 1_\square) = 1_\square, \quad \mathfrak{I}(1_\square, 0_\square) = 0_\square,$$

and such that $\mathfrak{I}(., x)$ is decreasing and $\mathfrak{I}(x, .)$ increasing, for all $x \in L_\square$.

The notions of S-implicator and R-implicator on $L_\square$ are defined in a similar way as in L^*. Unlike in L^* the R-implicator generated by the t-representable extension $\mathfrak{T}_W$ of T_W to $L_\square$ is equal to the S-implicator generated by $\mathfrak{S}_W$ and $\mathfrak{N}_s^2$:

$$\mathfrak{I}_{\mathfrak{T}_W}(x,y) = \mathfrak{I}_{\mathfrak{S}_W,\mathfrak{N}_s^2}(x,y)$$
$$= (\min(1, y_1 + 1 - x_1), \max(0, y_2 - x_2)).$$

This implicator is however not equal to the S-implicator generated by $\mathfrak{S}_W$ and $\mathfrak{N}_s^1$, which is given by, for $x, y \in L_\square$, $\mathfrak{I}_{\mathfrak{S}_W,\mathfrak{N}_s^1}(x,y) = (\min(1, x_2 + y_1), \max(0, x_1 + y_2 - 1))$.

6 The residuation principle

We say that a t-norm $\mathcal{T}$ on L^* satisfies the residuation principle if and only if, for all $x, y, z \in L^*$,

$$\mathcal{T}(x,z) \leq_{L^*} y \Leftrightarrow z \leq_{L^*} \mathcal{I}_\mathcal{T}(x,y).$$

The residuation principle for t-norms on $L_\square$ can be introduced in a similar way. For instance, the t-norms inf, $\mathcal{T}_w$, $\mathcal{T}_W$ and $\mathfrak{T}_W$ all satisfy the residuation principle.

De Baets and Mesiar proved in [7] that if a t-norm T on a complete lattice $L = L_1 \times L_2$ satisfies the residuation principle, then T is the direct product of two t-norms on L_1 and L_2 respectively. This result can be translated in our terminology as follows.

Theorem 6.1 *Any t-norm $\mathfrak{T}$ on $L_\Box$ satisfying the residuation principle is t-representable.*

Note that this result does not hold in L^*: $\mathcal{T}_W$ satisfies the residuation principle but is not t-representable!

7 Axioms of Smets and Magrez

The suitability of implicators for a variety of purposes can be assessed using the criteria introduced by Smets and Magrez in [12]. In [6] these axioms have been generalized to L^* as follows.

Definition 7.1 (Axioms of Smets and Magrez for an implicator $\mathcal{I}$ on L^*)

(A.1) $(\forall y \in L^*)(\mathcal{I}(., y)$ *is decreasing)* $(\forall x \in L^*)(\mathcal{I}(x, .)$ *is increasing)* (monotonicity laws)

(A.2) $(\forall x \in L^*)(\mathcal{I}(1_{L^*}, x) = x)$ (neutrality principle)

(A.3) $(\forall (x, y) \in (L^*)^2)(\mathcal{I}(x, y) = \mathcal{I}(\mathcal{N}_\mathcal{I}(y), \mathcal{N}_\mathcal{I}(x))$ (contrapositivity)

(A.4) $(\forall (x, y, z) \in (L^*)^3)(\mathcal{I}(x, \mathcal{I}(y, z)) = \mathcal{I}(y, \mathcal{I}(x, z)))$ (interchangeability principle)

(A.5) $(\forall (x, y) \in (L^*)^2)(x \leq_{L^*} y \Leftrightarrow \mathcal{I}(x, y) = 1_{L^*})$ (confinement principle)

(A.6) $\mathcal{I}$ *is a continuous* $(L^*)^2 \to L^*$ *mapping* (continuity)

In (A.3) $\mathcal{N}_\mathcal{I}$ denotes the negator on L^* induced by $\mathcal{I}$, defined as $\mathcal{N}_\mathcal{I}(x) = \mathcal{I}(x, 0_{L^*})$, for all $x \in L^*$.

The Smets-Magrez axioms for implicators on $L_\Box$ are introduced in a similar way. In [6] it is proven that $\mathcal{I}_{\mathcal{T}_W}$ satisfies all six Smets-Magrez axioms. Furthermore no S-implicator generated by a t-representable t-conorm and no R-implicator generated by a t-representable t-norm satisfies all six axioms. On the other hand, in $L_\Box$ we have that $\mathfrak{I}_{\mathfrak{T}_W}$ satisfies all Smets-Magrez axioms (and $\mathfrak{I}_{\mathfrak{S}_W, \mathfrak{N}_s^1}$ does not). In other words, t-representability plays very different roles in L^* and $L_\Box$.

8 Conclusion

Intuitionistic fuzzy logic and fuzzy four-valued fuzzy logic are closely related from a semantical point of view. However the underlying mathematical structures (respectively the "triangle" L^* and the "square" $L_\Box$) are not order-isomorphic. We constructed a representation for involutive negators on $L_\Box$ and showed the differences with the representation for involutive negators on L^*. We extended the Łukasiewicz t-norm to a t-representable t-norm on $L_\Box$ which satisfies similar properties. On the other hand the extension of the Łukasiewicz t-norm to L^* which satisfies similar properties is not t-representable. No residual implicator generated by a t-representable t-norm on L^* satisfies all Smets-Magrez axioms, but on $L_\Box$ there exists a t-representable t-norm whose residual implicator does. Finally we showed that there does not exist a Kleene negator on L^*, but there exists one on $L_\Box$. These observations confirm that L^* and $L_\Box$ are totally different structures from the mathematical point of view.

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