

Intuitionistic Fuzzy Sets and Interval-Valued Fuzzy Sets: a Critical Comparison

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Abstract

We confront two models that extend Zadeh's fuzzy set theory: intuitionistic fuzzy set theory and interval-valued fuzzy set theory. Our exposition recalls the syntactical relationships, in terms of L -fuzzy sets, that link these extensions and articulates the semantical factors that distinguish them.

Keywords: intuitionistic fuzzy sets, interval-valued fuzzy sets, L -fuzzy sets, type 2 fuzzy sets, bilattices

1 Some Background

Since its introduction in the sixties [23], fuzzy set theory has rapidly acquired an immense popularity as a formalism for the representation of imprecise, linguistic information: a vague concept is described by a membership function attributing to all elements of a given universe X a degree of membership in a fuzzy set. Since membership functions map X to the interval $[0, 1]$ and therefore imply a linear, i.e. total, ordering of these elements, one could argue that this makes them inadequate to deal with incomparable information. A possible solution, however, was already implicit in Zadeh's seminal paper [23]: in a footnote, he mentioned that *"in a more general setting, the range of the membership function can be taken to be a suitable partially ordered set P ."* In 1967, Goguen [14] formally introduced the notion of an L -fuzzy set with a membership function taking values in a lattice L .

Interval-valued fuzzy sets (IVFSs, for short), apparently first studied by Sambuc [20] who called them Φ -flou functions, serve to capture a feature of uncertainty w.r.t. the assignment of membership degrees. While many interpretations exist [17], the central idea is to replace crisp, $[0, 1]$ -valued membership degrees by intervals in $[0, 1]$, understood to contain the true, incompletely known membership degree.¹ In the words of Gehrke et al. [12], *"Many people believe that assigning an exact number to an expert's opinion is too restrictive, and that the assignment of an interval of values is more realistic."* There is an interesting parallel with imprecise probability theory, where crisp probabilities are replaced by intervals bounded by an upper and a lower probability (see e.g. Walley [22]).

Finally, intuitionistic fuzzy sets (IFSs, for short) were introduced in 1983 by Atanassov [1]. IFS theory basically defies the claim that from the fact that an element x "belongs" to a given degree (say $\mu_A(x)$) to a fuzzy set A , naturally follows that x should "not belong" to A to the extent $1 - \mu_A(x)$, an assertion implicit in the concept of a fuzzy set. On the contrary, IFSs assign to each element of the universe both a degree of membership $\mu_A(x)$ and one of non-membership $\nu_A(x)$ such that $\mu_A(x) + \nu_A(x) \leq 1$, thus relaxing the enforced duality $\nu_A(x) = 1 - \mu_A(x)$ from fuzzy set theory. Imagine, for instance, a voting procedure in which delegates have to express their feelings w.r.t. a number of proposals. It is obvious that while one can be in favour or in disfavour of a proposal to a certain extent, one can also abstain

¹As a consequence, the membership function μ is replaced by a couple (μ_l, μ_u) such that $\mu_l \subseteq \mu_u$.

(partially or totally) from the vote; an attitude inspired by, e.g., a lack of background or interest, or simply because no obvious arguments for or against the cause at stake have been raised. In such a situation, using only a $[0, 1]$ -valued degree expressing support for the proposal is arguably too committing, and we should be duly hesitant to classify him as a supporter or an opponent of the proposal. IFSs circumvent this problem by allowing one to address the positive and the negative side of an imprecise concept separately, and by not insisting that these assessments be exactly complementary [6].

2 Syntactical Relationships

In this section, we recall some known results from [8]. Let (L, \leq_L) be a complete lattice. Then Goguen [14] defined an L -fuzzy set in X as an $X \rightarrow L$ mapping. Putting e.g. $L^* = \{(x_1, x_2) \in [0, 1]^2 \mid x_1 + x_2 \leq 1\}$, $(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, one can easily verify that (L^*, \leq_{L^*}) is a complete lattice, and also that the class of IFSs is isomorphic to that of L^* -fuzzy sets. Hence, IFSs emerge, syntactically, as a specific subclass of L -fuzzy sets. A graphical representation of L^* , called **intuitionistic fuzzy interpretation triangle**, is shown in figure 1.

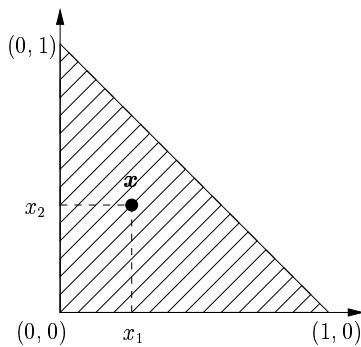


Figure 1: The lattice L^*

On the other hand, defining (L^I, \leq_{L^I}) as $L^I = \{(x_1, x_2) \in [0, 1]^2 \mid x_1 \leq x_2\}$, $(x_1, x_2) \leq_{L^I} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \leq y_2$, it is clear that IVFSs are also specific kinds of L -fuzzy sets. Moreover, IFSs are formally equivalent to IVFSs: indeed, a couple $(x_1, x_2) \in L^*$ may be mapped bijectively to an interval $[x_1, 1 - x_2]$. Figure 2 juxta-

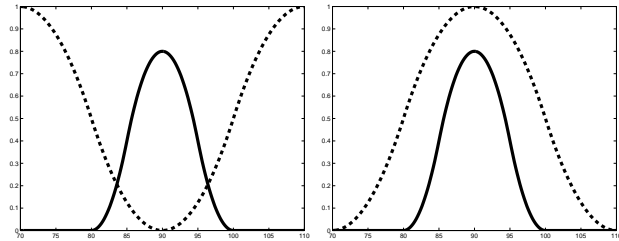


Figure 2: Geometrical Interpretation of a) an IFS and b) an IVFS .

poses the common geometrical interpretations [3] of an (equivalent) IFS and IVFS.

3 A Case for Semantical Differentiation

Section 2 highlighted that IFSs and IVFSs, when traced back to the underlying mathematical structure they are defined on, collapse to the same syntactical entity. Some would consider this equivalence sufficient evidence to dismiss IFS theory as superfluous and giving cause to unnecessary confusion. We wish to counter that allegation by demonstrating that the theories are at the crossroads of two important, different traditions.

The late George Gargov, accredited with giving intuitionistic fuzzy sets their name, argued in his last paper [11] that two ways exist to account for the uncertainty of knowledge²:

1. to admit either partial or contradictory models, or both
2. to consider sets of truth values (in particular, intervals) as representatives of the temporarily unknown truth value of a statement

3.1 Partial and/or Contradictory Models

The first trend is represented, in a very general way, by the notion of a bilattice (due to Ginsberg [13]). We introduce some terminology adapted from Fitting [9]: a pre-bilattice is a structure (B, \leq_t, \leq_k) where B is a non-empty set,

²Gargov referred explicitly to a logical setting, but it is not difficult to extend his argument to concern imprecision in general.

and \leq_t and \leq_k are partial orders on B each generating on it the structure of a lattice.

These orders are used to provide *methods of evaluation of information*. According to [11], information about the state of affairs described or referred to by a statement (like a membership degree assignment), can be characterized in two ways: by a degree of truth (cf. \leq_t), reflecting the truth content of the statement, and by a degree of knowledge (cf. \leq_k), reflecting information definiteness.

Denote the meet and join operations w.r.t. \leq_t as \wedge_t and \vee_t , and those w.r.t. \leq_k as \wedge_k and \vee_k . (B, \leq_t, \leq_k) is a bilattice³ if each of $\wedge_t, \vee_t, \wedge_k$ and \vee_k is monotone w.r.t. both orderings. The intended meaning of these requirements is that we insist e.g. that the truth content of a union of information about two statements does not decrease with the increase of knowledge. [11] A bilattice is distributive if all twelve possible distributive laws involving $\wedge_t, \vee_t, \wedge_k$ and \vee_k hold.

An example of a distributive bilattice is $([0, 1]^2, \leq_t, \leq_k)$, where $(x_1, y_1) \leq_t (x_2, y_2) \iff x_1 \leq x_2$ and $y_1 \geq y_2$, and $(x_1, y_1) \leq_k (x_2, y_2) \iff x_1 \leq x_2$ and $y_1 \leq y_2$. The first component gives a positive degree (of truth, or membership), while the second gives a negative degree (of falsity, or non-membership). By defining the correspondences $T \rightarrow (1, 0)$, $F \rightarrow (0, 1)$, $U \rightarrow (0, 0)$ and $C \rightarrow (1, 1)$, this structure also emerges as the “fuzzified” version of Belnap’s four-valued paraconsistent logic [5], with truth values **true** (T), **false** (F), **unknown** (U) and **contradiction** (C), shown in figure 3.

The structure has proven very relevant in e.g. decision-making problems; first steps to account for preferential information under the form of positive and negative arguments with graded relevance were taken in [19] and [10]. It is obviously also related to the framework of IFS theory, both syntactically and in terms of intended semantics. Specifically, we may define an operation called *conflation* on a bilattice, i.e. an involutive $B \rightarrow B$ mapping \mathcal{C} that reverses the \leq_k ordering but leaves the \leq_t ordering unchanged. We say that $x \in B$ is *consistent* if $x \leq_k \mathcal{C}(x)$ and that x

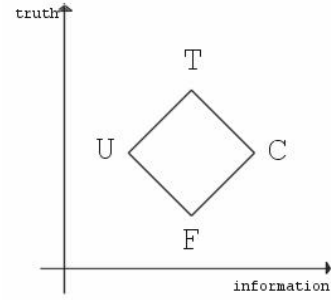


Figure 3: Belnap’s bilattice

is *exact* if $x = \mathcal{C}(x)$. It can easily be seen that if $\mathcal{C}(x_1, x_2) = (1 - x_2, 1 - x_1)$, L^* is the set of consistent elements of $([0, 1]^2, \leq_1, \leq_2)$, while $[0, 1]$ is the set of exact elements. Elements $(x_1, x_2) \in [0, 1]^2$ such that $x_1 + x_2 > 1$ (which in other words are *inconsistent* to some extent) are not considered in intuitionistic fuzzy set theory.

Incidentally, having in mind the fact from [4] that we can construct a bijection between the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$, and the IFS interpretation triangle L^* , we may assert that each L -fuzzy set with a lattice L that can be represented in the form of figure 3, can be represented as an IFS, too.

From a philosophical point of view, Smarandache’s promising new theory (with ancient roots) of neutrosophy [21] embraces a similar fundamental point of tripartition in sampling information: truth, falsity and indeterminacy have to be addressed. It even surpasses the bilattice setting because no relationship is required in general between these three components. In Smarandache’s words: “Every idea $\langle A \rangle$ tends to be neutralized, diminished, balanced by $\langle Non-A \rangle$ ideas (not only $\langle Anti-A \rangle$ as Hegel asserted)—as a state of equilibrium.” We expect that these ideas, once properly formalized, will have a profound impact on our future dealings with imprecision.

3.2 Intervals and Beyond

Section 1 already mentioned several authors’ disapproval of the practice of assigning crisp membership degrees to define an imprecise concept. To

³Unfortunately, neither Gargov nor Fitting gave a rigid definition of a bilattice. What we call a bilattice here is referred to as an “interlaced bilattice” by those authors.

fix these degrees only partially, e.g. by means of an interval or, to stick with Gargov, by arbitrary subsets of the evaluation space, alleviates that problem. Unsurprisingly, however, critics may argue that assessing e.g. exact interval borders is hard to justify. Nguyen et al. [17] suggested a notion of intervals “with *intervalsly uncertain bounds*”, i.e. the borders of the intervals are themselves characterized by intervals. To go one step further is to allow fuzzy sets in $[0, 1]$ as membership degrees: this is the setting of type 2 fuzzy set theory (see e.g. [15]). In general, *higher-order fuzziness* allows us to be more and more “fuzzy” about the assignment of membership degrees, a practice that has been likened to assessing the moments of a probability density function [16].

3.3 At the Crossroads of Imprecision—Which Way to Go?

As argued in the preceding subsections, IFS and IVFS theory should ideally be associated with their respective traditions (“paradigms”) of bilattices (focusing on knowledge/truth interplay, and positive/negative balance) on one hand, and of pursuing higher-order fuzziness on the other hand. In practice, the line is not always drawn so rashly. Nguyen et al. [17] appear not to distinguish at all between the two concepts and, most strikingly perhaps, Gargov himself is not entirely strict in putting a label on IFS theory. According to us, the truth is, these theories are located at a crossroads of imprecision-handling strategies, a base from which to move on to even more challenging directions; lifting the restriction on the sum of membership and non-membership values to obtain the fuzzy four-valued framework of [10] is one option, the definition of interval-valued intuitionistic fuzzy sets [2] is another one (and provides at once a marvellous combination of all we have discussed so far). It is precisely by these extensions that the need for separate paradigms to express different facets of imprecise information becomes obvious. For instance, intervals are ill-suited to the representation of partially conflicting information, as e.g. writing $[0.5, 0.4]$ is something of an awkward notation as compared to the alternative notation as a couple, i.e. $(0.5, 0.6)$ —where the first and second component quote beliefs in favour and disfavour of a statement, respectively.

On the other hand, we do plead to strengthen ties between scientific communities to gain deeper understanding of our joint goal, which is to process imprecise information efficiently. On the syntactical level, this means abstraction from outward appearances; the approach in terms of L -fuzzy sets advocated in section 2 seems cut out for the job. As an example, we may quote the definition of logical implication in the setting of both IFS and IVFS theory which was the object of [7]. On the semantical level, a cross-fertilization of, and dialogue between, logic, artificial intelligence and philosophy will likely further our cause most.

4 Conclusion

The extensive research being done on both IFSs and IVFSs (a recent survey [18] lists over 400 publications in the domain of IFS theory alone, and the number is still growing fastly) shows a mounting interest in these models which unfortunately is not paralleled by an according amount of scientific recognition. This paper has attempted to mend that situation by promoting the exploitation of their syntactical equivalence via a treatment in terms of L -fuzzy sets, and by identifying some essential semantical differences in their underlying motivation, differences that become clearest when we try to find suitable extensions for them. For this reason, choosing to maintain both the IFS and IVFS labels hardly engenders unnecessary proliferation of terminology, as some critics claim.

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References

- [1] K. T. Atanassov (1983). Intuitionistic Fuzzy Sets, VII ITKR’s Session, Sofia, June 1983 (Deposited in Central Sci. - Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian).

- [2] K. T. Atanassov, G. Gargov (1989). Interval-Valued Intuitionistic Fuzzy Sets, *Fuzzy Sets and Systems* **31(3)**, pages 343–349.
- [3] K. T. Atanassov (1999). *Intuitionistic Fuzzy Sets*, Physica-Verlag, Heidelberg – New York.
- [4] K. T. Atanassov (2002). Remark on a Property of the Intuitionistic Fuzzy Interpretation Triangle, *Notes on Intuitionistic Fuzzy Sets* **8(1)**, pages 34–36.
- [5] N. Belnap (1977). A Useful Four-Valued Logic, *Modern Uses of Multiple-Valued Logics* (D. Reidel, ed.), pages 8–37.
- [6] C. Cornelis, M. De Cock, E. E. Kerre (2002). Assessing Degrees of Possibility and Certainty within an Unreliable Environment, *Proceedings of RASC 2002*, pages 194–199.
- [7] C. Cornelis, G. Deschrijver, E. E. Kerre (2002). Implication in Intuitionistic and Interval-Valued Fuzzy Set Theory: Construction, Classification, Application, Submitted to *International Journal of Approximate Reasoning*.
- [8] G. Deschrijver, E. E. Kerre (2003). On the Relationship between Some Extensions of Fuzzy Set Theory, *Fuzzy Sets and Systems* **133(2)**, pages 227–235.
- [9] M. Fitting (1989). Bilattices and the Theory of Truth, *Journal of Philosophical Logic* **18**, pages 225–256.
- [10] P. Fortemps, R. Slowinski (2002). A graded quadrivalent logic for ordinal preference modelling: Loyola-like approach, *Fuzzy Optimization and Decision Making* **1**, pages 93–111.
- [11] G. Gargov (1999). Knowledge, Uncertainty and Ignorance in Logic: Bilattices and Beyond, *Journal of Applied Non-Classical Logics* **9(2–3)**, pages 195–283.
- [12] M. Gehrke, C. Walker, E. Walker (1996). Some Comments on Interval-Valued Fuzzy Sets, *Int. Journal of Intelligent Systems* **11(10)**, pages 751–759.
- [13] M. L. Ginsberg (1988). Multivalued Logics: a Uniform Approach to Reasoning in Artificial Intelligence, *Computational Intelligence* **4**, pages 265–316.
- [14] J. Goguen (1967). L-fuzzy Sets, *J. Math. Anal. Appl.* **18**, pages 145–174.
- [15] R. John (1998). Type 2 Fuzzy Sets: an Appraisal of Theory and Applications, *Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems*, **6(6)**, pages 563–576.
- [16] J. M. Mendel (2001). *Uncertain rule-based fuzzy logic systems*, Prentice Hall PTR, Upper Saddle River, New Jersey.
- [17] H. T. Nguyen, V. Kreinovich, Q. Zuo (1997). Interval-Valued Degrees of Belief: Applications of Interval Computations to Expert Systems and Intelligent Control, *Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems*, **5(3)**, pages 317–358.
- [18] M. Nikolova, N. Nikolov, C. Cornelis, G. Deschrijver (2002). Survey of the research on intuitionistic fuzzy sets, *Advanced Studies in Contemporary Mathematics*, **4(2)**, pages 127–157.
- [19] P. Perny, A. Tsoukias (1998). On the Continuous Extension of a Four Valued Logic for Preference Modelling, *Proceedings of IPMU 1998*, pages 302–309.
- [20] R. Sambuc (1975). *Fonctions Φ -floues. Application l'Aide au Diagnostic en Pathologie Thyroïdienne*, Ph. D. Thesis, Univ. Marseille, France.
- [21] F. Smarandache (2002). Neutrosophy, a New Branch of Philosophy, *Multiple Valued Logic* **8(3)**, pages 297–384.
- [22] P. Walley (1991). *Statistical Reasoning with Imprecise Probabilities*, Chapman and Hall.
- [23] L. A. Zadeh (1965). Fuzzy sets, *Information and Control* **8**, pages 338–353.