Assessing Degrees of Possibility and Certainty within an Unreliable Environment

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Abstracts: *Possibility distributions in Zadeh's sense restrict the possible values that a variable may assume; however, they do not leave any room for representing either reliability of the observer that provided the information, or hesitation, on the part of the observer, in fixing the exact possibility degrees. We show how Atanassov's intuitionistic fuzzy set theory can be used to mend each of these problems in a flexible and transparent manner.*

Keywords: possibility theory, degree of possibility, degree of certainty, intuitionistic fuzzy sets, hesitation, reliability

1 Introduction and Preliminaries

1.1 The Classical Account of Possibility Theory

Possibility theory in the sense of Zadeh [6] evolves around the notion of a so–called *elastic restriction* that allows us to discriminate between the more or less possible (plausible) values for a variable X in a universe U; in other words, it reflects our *uncertainty about the true value of* X. This elastic restriction is modelled by a mapping π_X from U to $[0, 1]$, called possibility distribution, such that $\pi_X(u) = p$ means that it is *possible* to degree p that X takes the value u. It is assumed that there exists at least one value $u \in U$ such that $\pi_X(u) = 1$; this is called a **normalization constraint**.

Example 1.1 (Kurt's length) *When we asked Kurt, a lively five–year old boy, to describe his length, he told us "I'm now almost one meter high". Taking his statement for granted, we modelled his length by a symmetrical possibility distribution on the universe of positive real numbers (representing length in centimeters) centered around the value of 95. The situation is depicted in figure 1.*

Possibility distributions occur frequently in the framework of approximate reasoning, where they represent linguistic information in a variety of knowledge–handling systems; e.g. they are used to model statements as well as rules in fuzzy expert systems, to describe incomplete or imprecise values for data attributes and to express soft constraints in fuzzy databases, . . . It is crucial to distinguish them from probability distributions[1]: the latter describe uncertainty caused by physical randomness, leading to questions like

"Given any five–year–old, what are the odds that his or her length exceeds one meter?" (1)

[1]For a very comprehensive and clear account of their mutual differences, we refer to the table on page 204 of [5].

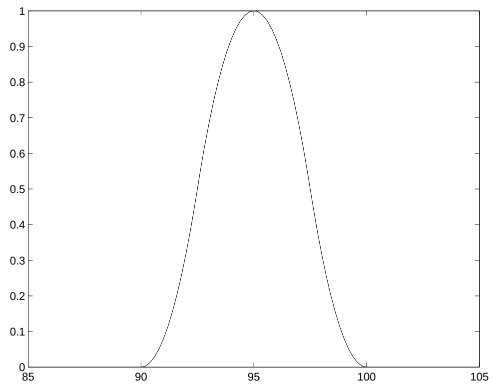

Figure 1: Possibility distribution for Kurt's length

while in possibility theory our uncertainty is due to a lack of knowledge about the exact value of an otherwise precisely defined quantity, giving way to questions like

$$\text{"Given all we know about Kurt, to what extent is it } \textit{plausible} \text{ to assume that his length exceeds one meter, and, to what extent can we be } \textit{certain} \text{ of that?"} \quad (2)$$

In (2), the occurrence of a twofold question highlights another important feature of possibility theory, again distinguishing it from probability theory: while the probabilities are taken to be self–evident and needing no further explanation, possibility theory is concerned with two related but essentially different types of knowledge. Indeed, we might want to assess to which degree it is *plausible* (i.e., not contradicted by the available evidence) that Kurt is taller than 1 meter, but in some cases we would like to have a quantitative measurement of our *certainty* about the statement, e.g. before making a commitment. This distinction is catered for in possibility theory by additionally associating with each variable X a necessity distribution n_X, such that $n_X(u) = n$ means that it is *certain* to degree n that $X = u$.

Evidently, possibility and necessity distributions should behave according to some preset intuitive criteria, as for instance it makes no sense to assert that $X = u$ is more certain than possible. Possibility theory heeds inconsistencies like these by setting

$$n_X(u) = 1 - \sup_{v \neq u} \pi_X(v) = \inf_{v \neq u}(1 - \pi_X(v)) \quad (3)$$

expressing that we are certain of $X = u$ only to the extent to which the remaining alternatives $X = v$ are not plausible. As can be seen, it suffices to provide either one of the possibility or necessity distributions to fully characterize an elastic restriction; in other words, the elastic restriction formally behaves as a (normalized) fuzzy set in U. We call (3) the **possibility/necessity duality constraint**.

Finally, to allow us to answer questions like (2) that concern sets of values, possibility theory introduces two dual measures Π_X and N_X of possibility and necessity, defined for any set $A \in \mathcal{P}(U)$ by

$$\Pi_X(A) = \sup_{u \in A} \pi_X(u) \quad (4)$$

$$N_X(A) = \inf_{u \notin A} 1 - \pi_X(u) \quad (5)$$

They are linked by the equality

$$N_X(A) = 1 - \Pi_X(co(A)) \quad (6)$$

where $co(A)$ denotes the complement of the subset A of U.

1.2 The Intuitionistic Fuzzy Account of Possibility Theory

In [2], we argued that the duality constraint (3) is overly restrictive and naive, since it forces us into a position of unconditional faith in the observer that assessed the degrees of possibility/necessity. For instance, when the observer claims that $\pi_X(u) = 0$ (u is an impossible value for X), by equation (5) with $A = co(\{u\})$ we are trapped into accepting with complete certainty that the true value of X is different from u, *with no room whatsoever for discredit in the observation skills or truthfulness of the information source.* Guided by the commonsense principle that certainty is a much stronger and decisive kind of knowledge than possibility and must be dealt with appropriately, in [2] we opted not to let the former be solely determined by the latter: we introduced an additional $U \to [0,1]$ mapping ν_X, such that $\nu_X(u)$ represents the certainty that X differs from u, linked to π_X by a simple constraint: $\pi_X(u) + \nu_X(u) \leq 1$ holds for every u, so formally these two distributions which lay down the elastic restriction behave like the membership and non–membership function[2] of an intuitionistic fuzzy set [1] in U. For this reason, the couple (π_X, ν_X) was called an intuitionistic fuzzy possibility distribution (IFPD) in [2]. Within this framework, moreover, redefining

$$n_X(u) = \inf_{v \neq u} \nu_X(v), \tag{7}$$

we obtain

$$n_X(u) \leq 1 - \sup_{v \neq u} \pi_X(v), \tag{8}$$

a weakened version of (3); the equality is obtained if $\nu_X(u) = 1 - \pi_X(u)$ for all $u \in U$ (complete belief in the observer, i.e. the case of classical possibility theory). In consequence, a low value of $\pi_X(u)$ will not commit us to very drastic conclusions if $\nu_X(u)$ is low as well.

Furthermore, in [2] we proposed altered definitions for certainty and possibility measures associated to an IFPD π_X:

$$\Pi_X(A) = \sup_{u \in A} \pi_X(u) \tag{9}$$

$$N_X(A) = \inf_{u \notin A} \nu_X(u) \tag{10}$$

Example 1.2 (Kurt's length, continued) *When we laid down the possibility distribution for Kurt's length in example 1.1, we relied solely on the boy's claim. Yet, as our experience reveals that young children tend to exaggerate about themselves, we might take his statement into doubt, conjecturing that his real length will be* considerably *smaller than 1 meter, instead of just* a bit *smaller. In order to express that skepticism, we represent his statement as the IFPD depicted in figure 2.*

1.3 A New Direction: The L^*–Valued Account of Possibility Theory

This paper complements the proposal outlined above with an additional extension constituting—in some sense—a more radical departure from the original notion of possibility theory; specifically, we will abandon the convention that degrees of possibility and necessity are evaluated in

[2]Let $(L^*, \leq_{L^*})$ be the complete, bounded lattice defined by [3]:

$$L^* = \{(x_1, x_2) \in [0,1]^2 \mid x_1 + x_2 \leq 1\}$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2$$

The units of this lattice are denoted $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. An intuitionistic fuzzy set (IFS) A in a universe U is a mapping from U to L^*, i.e. a special kind of L–fuzzy sets in the sense of Goguen [4]. The first component $(A(u))_1$ of $A(u)$ is called the membership degree of u to A, while the second component $(A(u))_2$ is called the non–membership degree of u to A. The complement $co(A)$ of an IFS A is defined by, for $u \in U$, $co(A)(u) = ((A(u))_2, (A(u))_1)$

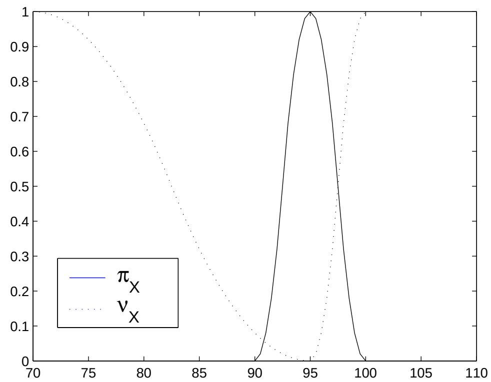

Figure 2: Intuitionistic fuzzy possibility distribution for Kurt's length

$[0, 1]$. Rather, we will evaluate them in L^*, the evaluation set of intuitionistic fuzzy set theory, thus acknowledging that an additional degree of hesitation or uncertainty may arise in their assessment. As will be shown, the envisaged extension is co–existent and not conflicting with the existing proposal from [2].

2 An Extended Possibility/Necessity Evaluation Framework

2.1 From $[0, 1]$–Valued Degrees to L^*–Valued Degrees

The essence of intuitionistic fuzzy sets lies in their acknowledgement that very often people, when assessing a degree, be it of membership, of truthhood, of possibility or of necessity, are reluctant to pin down that degree decisively, because they are to some extent hesitant about such an assignment (which would involve a strong commitment); what they often are prepared to do, is to fix a threshold α which they consider a **sure** lower bound for it, and to do the same for the reverse assessment problem (e.g. a sure degree β of non–membership, of falsehood, of impossibility or of uncertainty). Expressed in a different way, they approach the real value from below (by giving positive evidence α) and from above (by giving negative evidence β); the length of the resulting interval $[\alpha, 1 - \beta]$ is proportional to a person's hesitation.

Formally, this view gives way to the following definition: by an L^*–valued possibility distribution $\pi_X = (\pi_X^1, \pi_X^2)$ we denote any mapping from the universe U of X to L^*. π_X^1 will be called the **positive** possibility distribution of X while π_X^2 denotes the **negative** possibility distribution of X. We will illustrate this extension by the following example.[3]

Example 2.1 (Kurt's mum's perspective) *This time we asked Kurt's mum to tell us about her son's length. "Well," she says, "Last time we measured him he was 84cm but then that was three months ago… And they grow so quickly at that age!" We took somewhat more pains to come up with a realistic approximation of the possibility distribution, and therefore we asked the lady to produce positive and negative possibility degrees for lengths 85, 86, … cm and so on to obtain the following table.*

length	*85*	*86*	*87*	*88*	*89*	*90*	*91*	*92*	*93*	*94*	*95*
positive PD	*1*	*1*	*1*	*0.8*	*0.6*	*0.4*	*0.3*	*0.1*	*0*	*0*	*0*
negative PD	*0*	*0*	*0*	*0*	*0.1*	*0.3*	*0.3*	*0.3*	*0.4*	*0.7*	*1*

[3]Allowing the possibility degrees to be in L^*, we should naturally also draw necessity degrees $(n_X^1(u), n_X^2(u))$ from L^*. Equation (3) may be rewritten as: $(n_X^1(u), n_X^2(u)) = \sup_{v \neq u} co(\pi_X^1, \pi_X^2)(v)$.

Lengths between 85 and 87cm are perfectly possible in Kurt's mum's opinion. At 88cm she starts doubting; it's not that she'd be extremely surprised if this did turn out to be his real length, but somehow it seems a bit too optimistic. Her assignments gradually go down (observe that the resulting L^*–valued possibility distribution is *decreasing on* $[85, 95]$*), up to the point that at a length of 93cm, she frankly doesn't consider it a reasonable guess anymore, yet doesn't want to fully rule out an amazing growth spurt on Kurt's and incompetent measuring skills on her part.*

Observe the difference with example 1.2. There, acting as external referees, by the introduction of the distribution ν_X, we casted doubt about precise possibility degrees procured by an unreliable information source. In this example, however, the observer herself is not entirely sure and weakens her own claim by introducing a positive and a negative possibility distribution π_X^1 and π_X^2 resulting in a hesitation margin. Furthermore, Kurt's mum's hesitation concerned the assignment of possibility degrees, while we left Kurt's own possibility degrees unharmed only to act on their complement, a decision aimed towards modifying certainty. Naturally the question arises whether the two approaches could not be combined, i.e. we want to create an opportunity for introducing skepticism about the L^*–valued possibility degrees, and moreover we want to express skepticism (as opposed to hesitation) itself by means of values in L^*. This is discussed in the next paragraph.

2.2 L^*–Valued Intuitionistic Fuzzy Possibility Distributions

Sections 1.2 and 2.1 showed two alternative ways to apply the ideas of intuitionistic fuzzy set theory to Zadeh's possibility distributions. In this section, our aim is to merge these proposals and show how they can be used to express various facets of uncertainty in the representation of an unreliable observer's claims.

Let us first come back to our running example.

Example 2.2 (Kurt's length, conclusion) *The IFPD from figure 2 prevented us from making any too drastic conclusions about Kurt's length (which was just as well given his mum's input), but we can hardly say we're satisfied with it, because some values (e.g. 95) have received unrealistically high possibility degrees. One way of solving the problem is to dismiss the toddler's possibility degrees altogether and rely on the information gathered in example 2.1 instead; however, that would somehow thwart our original plan, which was* to model an observer's claim as well as our beliefs about the truthfulness of the claim.

We pick up the thread from example 1.2 to proceed in the following way: the degrees $\pi_X(u)$ and $\nu_X(u)$ are conceptually expanded to values $(\pi_X^1(u), \pi_X^2(u)) = (\pi_X(u), 1-\pi_X(u))$ and $(\nu_X^1(u), \nu_X^2(u))$ $= (\nu_X(u), 1-\nu_X(u))$ in L^*; for those potential values of X which we fear/believe have been assigned too optimistic possibility degrees, we decrease the first component $\pi_X(u)$ to a value $\pi_X^1(u)$; the larger our skepticism the smaller we make it. The result will still be an element of L^*, and moreover

$$(\nu_X^1(u), \nu_X^2(u)) \leq_{L^*} (\pi_X^2(u), \pi_X^1(u)) \tag{11}$$

holds, a faithful extension of the inequality $\nu_X(u) \leq 1 - \pi_X(u)$.

Before formalizing the ideas of L^*–valued IFPD's in the most general sense, we should mention two important things:

- Although different semantics were attached to L^*–valued possibility degrees in our examples 2.1 and 2.2 (one reflecting an observer's hesitation, the other an external judge's skepticism), the underlying idea is the same: fixing the degrees exactly is *risky*, *dangerous*, *undesirable*, *unjustified*, ... so we want to handle them with **reservation**. The only mandatory precaution to take is that one should always specify whom the uncertainty is emerging from: the information source, or its recipient.

- In example 2.2, nothing prevented us from changing $(\nu_X^1(u), \nu_X^2(u))$ as well as long as (11) is preserved; it would reflect a margin of hesitation about our skepticism.

Definition 2.1 (Ingredients of L^*–valued intuitionistic fuzzy possibility theory) *Let X be a variable assuming values in U. With X are associated two $U \to L^*$ mappings $\pi_X = (\pi_X^1, \pi_X^2)$ and $\nu_X = (\nu_X^1, \nu_X^2)$ such that there exists at least one $u \in U$ such that $\pi_X(u) = 1_{L^*}$ and $\nu_X(u) \leq_{L^*} (co\pi_X)(u)$ holds for all $u \in U$. With π_X and ν_X are associated two measures Π_X and N_X, defined for any set $A \in \mathcal{P}(U)$ by*

$$\Pi_X(A) = \sup_{u \in A} \pi_X(u) = \left(\sup_{u \in A} \pi_X^1(u), \inf_{u \in A} \pi_X^2(u)\right) \tag{12}$$

$$N_X(A) = \inf_{u \notin A} \nu_X(u) = \left(\inf_{u \notin A} \nu_X^1(u), \sup_{u \notin A} \nu_X^2(u)\right) \tag{13}$$

3 Conclusion

Intuitionistic fuzzy sets allow to generalize Zadeh's possibility theory in at least two, non–conflicting manners. In section 1.2 we proposed to break up the classical possibility–necessity duality by the introduction of a supplementary distribution ν_X; in section 2.1, we replaced $[0,1]$–valued possibility distributions by L^*–valued ones (consisting of a positive possibility distribution π_X^1 and a negative possibility distribution π_X^2). The resulting formalism can be used to express, in a quantitative manner, the reservation of the parties involved: hesitation on the part of the information source, and skepticism on the part of its recipient. Different schemes embodying this idea have been proposed in this paper.

Acknowledgements

Chris Cornelis and Martine De Cock would like to thank the Fund for Scientific Research Flanders (FWO) for funding the research elaborated on in this paper.

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