Representing Reliability and Hesitation in Possibility Theory: A General Framework

Chris Cornelis, Martine De Cock, and Etienne Kerre

Fuzziness and Uncertainty Modelling Research Unit, Applied Mathematics and Computer Science Dept., Krijgslaan 281 (S9), B–9000 Gent, Belgium
{chris.cornelis,martine.decock,etienne.kerre}@UGent.be
http://fuzzy.UGent.be

1 Introduction

Possibility theory, introduced by Zadeh in [5], is a formalism that tries to capture in mathematical terms imprecise (typically, linguistic) information about the more or less plausible values that a variable may assume. For instance, a statement like "decent houses in Gent do not come cheap" flexibly restricts plausible buying prices by pointing out that a low budget will not get very you far in this town. Compare this to "I found a nice place in Gent for about 100.000 EUR", which gives you a guarantee (by explicit example) that properties in a given price range can be found. The examples we quoted are instances of what is called negative or constraint–based and positive or observation–based information respectively in the literature [3, 4]: the first kind rules out certain values for the buying price X, while the second kind only designates certain observed values as "definitely possible" while saying nothing about the rest.

To mathematically represent both types of information, possibility distributions and guaranteed possibility distributions are introduced. Formally, a possibility distribution π_X on a variable X (e.g. buying price) in a universe U (e.g. a domain $[0, 200.000]$ of prices in euro) is a $U \to [0,1]$ mapping such that $\pi_X(u) = p$ means that it is (at most) possible to degree p that X can take the value u. Possibility degrees typically emerge by evaluating a fuzzy set like "expensive" and subsequently imposing

$$\pi_X(u) \leq expensive(u) \tag{1}$$

The inequality allows for the existence of other information items restricting X (specifically, new constraints can make the possibility degrees decrease). On the other hand, a guaranteed possibility distribution δ_X on U is another $U \to [0,1]$ mapping such that $\delta_X(u) = d$ means that it is (at least, or guaranteed) possible to degree d that X can take the value u. In our example, δ_X would be obtained by setting

$$\delta_X(u) \geq \text{about-100.000-EUR}(u) \tag{2}$$

Analogously to (1), the inequality is meant to imply that later on new observations can make the guaranteed possibility degrees increase. Constraint–based and observation–based information induce different types of inequalities; it is generally assumed that $\delta_X(u) \leq \pi_X(u)$ for $u \in U$, an integrity constraint expressing that impossible values cannot be guaranteed.

(Guaranteed) possibility distributions have been applied fruitfully in a wide range of knowledge–based systems. To increase their expressivity, it is worthwhile to take into account the following ideas:

- It is not always realistic to pin down exact (guaranteed) possibility degrees, since the person assessing them is hesitant about them to a given extent.
- We sometimes may have doubts about the truthfulness of information, i.e. claims can have varying degree of credibility.

In this paper, we study extensions, in terms of interval–valued fuzzy sets and intuitionistic fuzzy sets, of classical (guaranteed) possibility distributions that are able to deal with the facets described above. We will illustrate their usefulness by some concrete examples.

2 Preliminaries

2.1 Interval–Valued and Intuitionistic Fuzzy Sets

Interval–valued fuzzy sets (IVFSs, for short) form an extension of fuzzy sets in the sense that crisp, $[0,1]$–valued membership degrees (say, $\mu_A(u)$) are replaced by intervals $[\mu_A^l(u), \mu_A^u(u)]$ in $[0,1]$. As a general principle, IVFSs emerge whenever fixing exact membership degrees is conceived as unrealistic, unjustified or too restrictive.

Intuitionistic fuzzy sets (IFSs, for short) defy the claim that from the fact that an element u "belongs" to a given degree $\mu_A(u)$ to a fuzzy set A, naturally follows that u should "not belong" to A to the extent $1 - \mu_A(u)$, an assertion implicit in the concept of a fuzzy set. On the contrary, IFSs assign to each element of the universe both a degree of membership $\mu_A(u)$ and one of non–membership $\nu_A(u)$ such that $\mu_A(u) + \nu_A(u) \leq 1$, thus relaxing the enforced duality $\nu_A(u) = 1 - \mu_A(u)$ from fuzzy set theory.

It is instructive from a syntactical perspective to study IVFSs and IFSs as particular cases of L–fuzzy sets. To recall, let $(L, \leq_L)$ be a complete lattice. An L–fuzzy set in X is an $X \to L$ mapping. Putting e.g. $L^* = \{(x_1, x_2) \in [0,1]^2 \mid x_1 + x_2 \leq 1\}$, $(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, one can easily verify that $(L^*, \leq_{L^*})$ is a complete lattice, and also that the class of IFSs is isomorphic to that of L^*–fuzzy sets. Hence, IFSs emerge, syntactically, as a specific subclass of L–fuzzy sets. On the other hand, defining $(L_I, \leq_{L_I})$ as $L_I = \{(x_1, x_2) \in [0,1]^2 \mid x_1 \leq x_2\}$, $(x_1, x_2) \leq_{L_I} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and

$x_2 \leq y_2$, it is clear that IVFSs are also specific kinds of L–fuzzy sets. Moreover, IFSs are formally equivalent to IVFSs: indeed, a couple $(x_1, x_2) \in L^*$ may be mapped bijectively to an interval $[x_1, 1 - x_2]$.

For a more detailed account of the semantical aspects of IVFSs and IFSs, we refer to [2]. What should be clear is that IVFSs and IFSs correspond to distinct paradigms: the former acknowledge the lack of justification for precise membership degrees, while the latter address positive (membership) and negative (non–membership) assessments separately.

2.2 Selected Notions from Possibility Theory

As explained in the introduction, the essence of (guaranteed) possibility distributions is that they provide us with information about the values a variable X may assume. In the constraint–based framework, this information comes in a negative guise, expressing which values u are more or less impossible for X; this is done by associating with X a $U \to [0, 1]$ mapping π_X called a possibility distribution on X. π_X can be used to evaluate to what extent it is possible that X takes a value in a crisp subset A of U. To this aim, we calculate $\Pi_X(A) = \sup_{u \in A} \pi_X(u)$, the possibility of A. On the other hand, we may also be interested to know to what extent we can be sure that X takes its values in A; $N_X(A) = \inf_{u \notin A} 1 - \pi_X(u)$ is called the certainty or necessity of A. It is clear that $N_X(A) = 1 - \Pi_X(coA)$; this equality is known as possibility–necessity duality.

In the observation–based framework information emerges as soon as some value u receives a strictly positive degree $\delta_X(u)$ of guaranteed possibility, where the guaranteed possibility distribution δ_X on X is again a $U \to [0, 1]$ mapping δ_X. It can be used to define two dual measures Δ_X and ∇_X of guaranteed possibility and potential necessity. For a crisp subset A of U, $\Delta_X(A) = \inf_{u \in A} \delta_X(u)$ calculates the extent to which it is guaranteed possible that X takes a value in A, while $\nabla_X(A) = \sup_{u \notin A} 1 - \delta_X(u)$ expresses the potential ("best–case") necessity or certainty that X indeed takes its values in A. It can be verified that $\nabla_X(A) = 1 - \Delta_X(coA)$.

3 Extensions

In this section, we explore two kinds of situations which the traditional patterns of possibility theory have difficulties coping with. To each problem, an extension of the original definitions is proposed to mend their shortcomings.

3.1 Interval–Valued Extension

Recall the example statement from the introduction that houses in Gent are expensive. Compare it to the following conversation: "This house costs 250.000

EUR."— "Wow, that's expensive!" Whereas in the latter situation there is no uncertainty (the use of a linguistic term is just a way of abstracting from numerical data, in other words an instance of vagueness), in the former there is usually plenty; it seems therefore unnatural to suppress this uncertainty by enforcing the rigid delimitation that a fuzzy set representation of "expensive" implies. In other words, while people may be able to draft a membership function for "expensive" (evaluating prices on a given scale), when they assign the label to a given situation, usually this association is pervaded with some uncertainty. As a result, people may be reluctant to pin down exact (guaranteed) possibility degrees decisively, because they are to some extent hesitant about such an assignment (which would involve a strong commitment); what they often are prepared to do, is to fix thresholds α and β which they consider a sure lower and upper bound for it, respectively. In figure 1, this is illustrated for our example. The left hand side picture presents a π_X which is defined by the fuzzy set *expensive*, while in the right hand side picture, a margin of hesitation is introduced by introducing lower and upper possibility degrees $\pi_X^1(u)$ and $\pi_X^2(u)$.

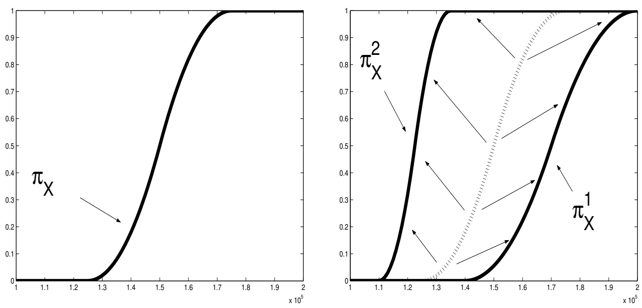

Fig. 1. Representation of hesitation. a) π_X is a fuzzy set b) π_X is an IVFS

Formally, by an interval–valued possibility distribution of a variable X on a universe U, we mean a mapping $\pi_X = [\pi_X^1, \pi_X^2]$ from U to L_I. The length of the interval $[\pi_X^1(u), \pi_X^2(u)]$ expresses, for each $u \in U$, the information source's hesitation w.r.t. the possibility assessment. From π_X, we may derive generalized versions of the possibility and necessity of a crisp subset A of U:

$$\begin{aligned} \Pi_X(A) &= \sup_{u\in A} \pi_X(u) &&= \left[\sup_{u\in A} \pi_X^1(u), \sup_{u\in A} \pi_X^2(u)\right] \\ N_X(A) &= \inf_{u\notin A} \mathcal{N}(\pi_X(u)) &&= \left[\inf_{u\notin A} 1-\pi_X^2(u), \inf_{u\notin A} 1-\pi_X^1(u)\right] \end{aligned}$$

where the complement operation $\mathcal{N}$ on L_I is defined by $\mathcal{N}([u_1, u_2]) = [1 - u_2, 1 - u_1]$. It can be verified that $N_X(A) = \mathcal{N}(\Pi_X(coA))$. Analogously, we can derive an interval–valued extension of guaranteed possibility distributions [3].

3.2 Intuitionistic Fuzzy Extension

With the traditional machinery of (guaranteed) possibility distributions, there is no way to cast doubt about a given statement. This becomes obvious when one takes a closer look at the relationship between (guaranteed) possibility and (potential) necessity. When $\pi_X(u) = 0$, then from the definition of necessity it is clear that $N_X(co\{u\}) = 1$, a statement that means that we are certain that X cannot take the value u. Unfortunately, this forces us into a position of unconditional faith in the truthfulness of the information provider. A similar remark holds when $\delta_X(u) = 1$ for some u; in that case, $\nabla_X(co\{u\}) = 0$, implying that the potential (and hence the actual) certainty that X cannot take the value u does not surpass 0. But what if we found out that whoever provided the information were lying? It should be a sign for us to cancel all commitments (i.e. assessments of non–trivial certainties) based on the information.

For this reason, we propose to opt out of the possibility–necessity duality (resp. the guaranteed possibility–potential necessity duality); to see how this can be done, recall the example "I found a place for about 100.000 EUR". The left–hand picture in figure 2 shows the corresponding guaranteed possibility distribution δ_X. Its complement (let us call it ω_X) is also shown; recall that $\omega_X(u) = \nabla_X(co\{u\})$ expresses, for $u \in U$, the potential certainty that X is different from u. Now let us assume that we got this information from someone who is always bragging about what good deals he has been able to strike, but in general pays much more than he is willing to admit. To account for this scepticism, we should indicate that the guaranteed possible degrees for certain low amounts of money are not actually credible in our eyes. We do this raising (increasing) ω_X, such that e.g. $\omega_X(100000) = 1$, i.e. the potential certainty that X differs from u is equal to one. This is shown in the right hand side picture of figure 2.

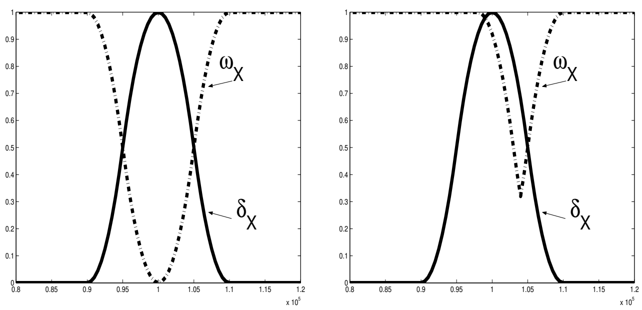

Fig. 2. Representation of unreliable information. a) δ_X and ω_X are each other's complement. b) δ_X and ω_X are partially independent distributions

Formally, by an intuitionistic fuzzy guaranteed possibility distribution of a variable X on a universe U, we mean a couple (δ_X, ω_X) where δ_X and ω_X are two $U \to [0,1]$ mappings such that $\delta_X(u) + \omega_X(u) \geq 1$ holds for all $u \in U$. Completely analogously, an intuitionistic fuzzy possibility distribution of a variable X on a universe U is a couple (π_X, ν_X) where π_X and ν_X are two $U \to [0,1]$ mappings such that $\pi_X(u) + \nu_X(u) \leq 1$ for all $u \in U$, which explains the terminology "intuitionistic fuzzy". The newly introduced concepts also give rise to altered definitions of (guaranteed) possibility and (potential) necessity measures. Given (π_X, ν_X) and (δ_X, ω_X), we may define, for a crisp subset A of U:

$$\begin{aligned} \Pi_X(A) &= \inf_{u \in A} \pi_X(u) & \Delta_X(A) &= \inf_{u \in A} \delta_X(u) \\ N_X(A) &= \sup_{u \notin A} \nu_X(u) & \nabla_X(A) &= \sup_{u \notin A} \omega_X(u) \end{aligned}$$

It is clear that $N_X(A) \leq 1 - \Pi_X(coA)$ and $\nabla_X(A) \geq 1 - \Delta_X(coA)$.

4 Conclusion

This paper described two extensions to the classical setting of possibility theory: an interval–valued extension that draws π_X and δ_X from L_I instead of from $[0,1]$, and an intuitionistic fuzzy extension that gives the concepts of (guaranteed) possibility and (potential) necessity (up to a certain point) independent status. The resulting formalism can be used to express, in a quantitative manner, the reservation of the parties involved: hesitation on the part of the information source, and skepticism on the part of its recipient. The material in this paper is exposed in a more detailed fashion in [3].

References

1. Cornelis C, De Cock M, Kerre E (2002) Assessing degrees of possibility and certainty within an unreliable environment. In: Lotfi A, Garibaldi J, John R (eds) Proceedings of Fourth International Conference on Recent Advances in Soft Computing.
2. Cornelis C, Atanassov K, Kerre E (2003) Intuitionistic Fuzzy Sets and Interval–Valued Fuzzy Sets: a Critical Comparison, To appear in: Proceedings of EUSFLAT 2003.
3. Cornelis C, Kerre E (2003) Generalized (Guaranteed) Possibility Distributions for Hesitation and Reliability Representation. Submitted to Fuzzy Sets and Systems.
4. Dubois D, Prade H, Ughetto L(2002) A New Perspective on Reasoning with Fuzzy Rules. In: Pal N, Sugeno M (eds) Proceedings of 2002 International Conference on Fuzzy Systems, LNAI 2275, Springer.
5. Zadeh L (1978) Fuzzy Sets as a Basis for a Theory of Possibility. Fuzzy Sets and Systems 1:3–28