Approximate Equality is no Fuzzy Equality

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Abstract

We argue that fuzzy equivalences, and in particular fuzzy equalities, in general are not suitable to model approximate equality due to the notion of transitivity. Using them for this purpose therefore leads to counter-intuitive results, as we illustrate in detail in a fuzzy relation-based framework for the representation of linguistic modifiers. We solve the problem by choosing resemblance relations.

Keywords: fuzzy equivalence, fuzzy equality, approximate equality, linguistic modifier, resemblance relation

1 Introduction

Approximate equality is a vague concept: the transition between *being* approximately equal and *not being* approximately equal is not abrupt but gradual. Therefore approximate equality on a universe X should be modelled by means a fuzzy relation on X, i.e. a fuzzy set on $X \times X$. It is natural to assume that this fuzzy relation should be reflexive and symmetrical. Furthermore we can take into account the following assumption:

Assumption 1 *The closer two objects are to each other (i.e. the smaller the distance between them), the more they are (approximately) equal.*

In [13] Zadeh has "fuzzified" the concept of a crisp equivalence relation into that of a similarity relation, i.e. a reflexive, symmetric and min-transitive fuzzy relation. Generalizations replacing min-transitivity by $\mathcal{T}$-transitivity for an arbitrary t-norm $\mathcal{T}$ appear in the literature under different names (e.g. [2, 10]), among which:

Definition 1 (Fuzzy $\mathcal{T}$-equivalence) *A fuzzy relation E on X is called a fuzzy $\mathcal{T}$-equivalence relation on X iff for all x, y and z in X:*

(FE.1) $E(x, x) = 1$
(FE.2) $E(x, y) = E(y, x)$
(FE.3) $\mathcal{T}(E(x, y), E(y, z)) \leq E(x, z)$

If the separation property holds, i.e. for all x and y in X it holds that $E(x, y) = 1$ iff $x = y$, then the fuzzy $\mathcal{T}$-equivalence E is called a **fuzzy $\mathcal{T}$-equality** [1]. Whenever it is clear, or not important, which specific t-norm is used, in this paper the terms "fuzzy equivalence" and "fuzzy equality" are used.

It is clear that because of the separation property, a fuzzy equality cannot express approximate equality in essence. In fact in this case two objects would be *approximately* equal to degree 1 *only* if they are *exactly* equal to degree 1. One can think of numerous examples in real life where this behaviour is undesired: e.g. we should be able to express that two temperatures 35° and 36° are approximately equal to degree 1, that two ages 34 years and 35 years are approximately equal to degree 1, that two prices 160 euro and 161 euro are approximately equal to degree 1, etc. although none of these values are exactly equal.

In the following section we will show that the notion of transitivity also leads to counter-intuitive results. Therefore not only fuzzy equalities but fuzzy equivalences in general are unsuitable to model approximate equality. This especially applies for similarity relations, although their name suggests the opposite. Throughout the paper let $\mathcal{F}(X)$ denote the class of all fuzzy sets on X. Let $\mathcal{T}$ denote a t-norm and $\mathcal{I}_{\mathcal{T}}$ its residual implicator, i.e. $\mathcal{I}_{\mathcal{T}}(x,y) = \sup\{\lambda |\ \lambda \in [0,1] \text{ and } \mathcal{T}(x,\lambda) \leq y\}$, for all x and y in $[0,1]$. Furthermore let $\mathcal{N}$ denote a negator and $\mathcal{T}^{\leftrightarrow\mathcal{N}}$ the dual of $\mathcal{T}$ w.r.t. $\mathcal{N}$, i.e. $\mathcal{T}^{\leftrightarrow\mathcal{N}}(x,y) = \mathcal{N}^{-1}(\mathcal{T}(\mathcal{N}(x),\mathcal{N}(y)))$, for all x and y in $[0,1]$. Recall that the dual of the minimum t-norm $\mathcal{T}_M$, the Łukasiewicz t-norm $\mathcal{T}_L$ and the drastic t-norm $\mathcal{T}_D$ w.r.t. the standard negator $\mathcal{N}_s$ are respectively the maximum t-conorm $\mathcal{S}_M$, the bounded sum $\mathcal{S}_L$ and the drastic sum $\mathcal{S}_D$. These operators are defined by, for all x and y in $[0,1]$, $N_s(x) = 1-x$, $\mathcal{T}_M(x,y) = \min(x,y)$, $\mathcal{T}_L(x,y) = \max(0, x+y-1)$, $\mathcal{T}_D(x,y) = \min(x,y)$ if $\max(x,y) = 1$ and $\mathcal{T}_D(x,y) = 0$ otherwise, $\mathcal{S}_M(x,y) = \max(x,y)$, $\mathcal{S}_L(x,y) = \min(1, x+y)$, and $\mathcal{S}_D(x,y) = \max(x,y)$ if $\min(x,y) = 0$ and $\mathcal{S}_D(x,y) = 1$ otherwise. Furthermore $\mathcal{I}_{\mathcal{T}_L}(x,y) = \min(1-x+y, 1)$. The complement of a fuzzy set A on X is defined as usual by $co_{\mathcal{N}}A(x) = \mathcal{N}(A(x))$, for all x in X.

2 Fuzzy equivalences

We want to stress that we do not question the concept of a fuzzy equivalence, but only its use to model approximate equality. First of all we take a look at the behaviour of fuzzy equivalences w.r.t. Assumption 1. The concept of a pseudometric reflects our intuitive understanding of the notion of distance.

Definition 2 (Pseudometric)[1] *A $X^2 - [0,1]$ mapping d is called a pseudometric on X iff for all x, y and z in X:*

(PM.1) $d(x,x) = 0$
(PM.2) $d(x,y) = d(y,x)$
(PM.3) $d(x,y) + d(y,z) \geq d(x,z)$

The couple (X,d) is called a pseudometric space.

The third condition (PM.3) can be replaced by one that involves a t-conorm $\mathcal{S}$, namely $\mathcal{S}(d(x,y), d(y,z)) \geq d(x,z)$ **(SPM.3)**, giving rise to the definition of a $\mathcal{S}$-pseudometric. Every $\mathcal{S}_1$-pseudometric on X is also a $\mathcal{S}_2$-pseudometric on X for $\mathcal{S}_1 \subseteq \mathcal{S}_2$. The condition imposed by $\mathcal{S}_M$-pseudometrics (i.e. pseudo-ultrametrics) is too severe from the intuitive point of view. Indeed, if y is some intermediate point between x and z, then $\mathcal{S}_M(d(x,y),\ d(y,z)) \geq d(x,z)$ does not correspond to our intuition. Leaving the subset of $\mathcal{S}_M$-pseudometrics and approaching the borderline of the class of $\mathcal{S}_L$-pseudometrics (i.e. the class of pseudometrics), this condition relaxes. Once we cross the border and leave the set of $\mathcal{S}_L$-pseudometrics however, the $\mathcal{S}$-pseudometrics are no longer pseudometrics. Note that for a $\mathcal{S}_D$-pseudometric, the condition **(SPM.3)** gives almost no information at all: if neither $d(x,y)$ nor $d(y,z)$ is 0, it corresponds to $1 \geq d(x,z)$ which is trivial considering the definition of d. Therefore in our opinion the best $\mathcal{S}$-pseudometrics to model distance are the pseudometrics in the neighbourhood of the $\mathcal{S}_L$-border. The links between fuzzy equivalences and $\mathcal{S}$-pseudometrics are well known:

Proposition 1 *[10] The following statements are equivalent for E a fuzzy relation on X:*

1. *E is a fuzzy $\mathcal{T}$-equivalence on X*
2. *$co_{\mathcal{N}}\ E$ is a $\mathcal{T}^{\leftrightarrow\mathcal{N}}$-pseudometric on X*

At first sight this seems to reflect the intuitive reverse connection between distance and approximate equality (Assumption 1) very good. An immediate result however is that E is a fuzzy M-equivalence on X iff $co_{\mathcal{N}_s}E$ is a pseudo-ultrametric on X. In other words Proposition 1 highlights the link between fuzzy $\mathcal{T}_M$-equivalences with a counter-intuitive concept to model distance! On the other hand E is a fuzzy $\mathcal{T}_L$-equivalence on X iff $co_{\mathcal{N}_s}E$ is a pseudometric on X, which indicates that fuzzy $\mathcal{T}_L$-equivalences (that are no $\mathcal{T}_M$-equivalences) are more natural. This is recognized by Valverde [10] and possibly explains the success of $\mathcal{T}_L$-equivalences in fuzzy databases [7].

But unfortunately fuzzy $\mathcal{T}_L$-equivalences are also not soul-saving in general: in [4] a detailed study is given of paradoxes arising when using a (arbitrary) fuzzy equivalence to model approximate equality. A typical example relates to the

[1] For simplicity we restrict ourself to $[0,1]$-valued pseudometrics. Note that every $[0,+\infty[$-valued pseudometric d' can be turned into a $[0,1]$-valued pseudometric d by defining $d(x,y) = \min(1, d'(x,y))$, for all x and y in X.

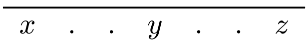

Figure 1: Intuitively: $E(x, y) = 1$, $E(y, z) = 1$ and $E(x, z) < 1$

Poincaré paradox [9]: suppose that there are three objects x, y and z such that x and y, and y and z resemble to degree 1, but x and z do not (see Figure 1 for an illustration). In such a case assigning a similarity degree 1 to the pairs (x, y) and (y, z) and applying $\mathcal{T}$-transitivity would imply that $E(x, z) = 1$ although we intuitively expect $E(x, z) < 1$. A similar problem is recognized in [12] where one tries to solve it by demanding **(FE.3)** only for some x, y and z.

Another interesting example stems from applying the compositional rule of inference: for v_1, v_2 and v_3 variables in X and E a fuzzy relation on X the CRI dictates [15]:

$$\begin{array}{l} (v_1, v_2) \text{ is } E \\ (v_2, v_3) \text{ is } E \\ \hline (v_1, v_3) \text{ is } E \circ_{\mathcal{T}} E \end{array}$$

in which $E \circ_{\mathcal{T}} E$ is the fuzzy relation on X defined by

$$E \circ_{\mathcal{T}} E(x, z) = \sup_{y \in X} \mathcal{T}(E(x, y), E(y, z))$$

It can be verified that for a reflexive and $\mathcal{T}$-transitive E it holds that $E \circ_{\mathcal{T}} E = E$. Hence, modelling approximate equality by a fuzzy $\mathcal{T}$-equivalence, the CRI would allow us to deduct from "The age of Jim (26) and the age of John (23) is approximately equal" and "The age of Jim (23) and the age of Mary (20) is approximately equal" the fact "The age of Jim (26) and the age of Mary (20) is approximately equal". By induction we would even be able to derive that all ages are approximately equal to each other.

The undesirability of $\mathcal{T}$-transitivity becomes also apparant in a fuzzy relation based framework for linguistic modifiers. For A a fuzzy set on X, R a fuzzy relation on X and y in X, the following representations are proposed in [5]:

$$\text{very } A\ (y) = R^{\heartsuit}(A)(y) = \inf_{x \in X} \mathcal{I}_{\mathcal{T}}(R(x, y), A(x))$$
$$\text{mol } A\ (y) = R^{\clubsuit}(A)(y) = \sup_{x \in X} \mathcal{T}(R(x, y), A(x))$$

(mol A corresponds to more or less A). They are based on the observation that y is called very A if every object x resembling to y is A. Likewise y is more or less A is y resembles to an x that is A. In both cases the concept of resemblance (approximate equality) is expressed by means of the fuzzy relation R. It can be verified that the operators $R^{\heartsuit}$ and $R^{\clubsuit}$ satisfy the following properties (see e.g. [3, 11]):

Proposition 2

1. *R is reflexive iff for all A in $\mathcal{F}(X)$*
$$R^{\heartsuit}(A) \subseteq A \subseteq R^{\clubsuit}(A)$$

2. *R is $\mathcal{T}$-transitive iff for all A in $\mathcal{F}(X)$*
$$R^{\heartsuit}(A) \subseteq R^{\heartsuit}(R^{\heartsuit}(A))$$
$$R^{\clubsuit}(R^{\clubsuit}(A)) \subseteq R^{\clubsuit}(A)$$

As we mentioned in the introduction, reflexivity is a natural characteristic of a fuzzy relation modelling approximate equality. In the framework of linguistic modifiers given above, this corresponds to very $A \subseteq A \subseteq$ more or less A. This so-called semantic entailment is often assumed in the litterature (see e.g. [8, 14]). Combined with the reflexivity, the $\mathcal{T}$-transitivity of R however will give rise to $R^{\heartsuit}(R^{\heartsuit}(A)) = R^{\heartsuit}(A)$ and $R^{\clubsuit}(R^{\clubsuit}(A)) = R^{\clubsuit}(A)$ which is clearly counter-intuitive: indeed "very very A" does not mean the same as "very A".

3 Resemblance relations

In [4] a new framework for modelling approximate equality is presented, namely the concept of a pseudo-metric resemblance relation. In this paper[2] we will assume that the universe can be equipped with a meaningful pseudo-metric d.

Definition 3 (Resemblance relation) *For (X, d) a pseudometric space, a fuzzy relation E on X is called a resemblance relation on X iff for all x, y, z and u in X:*

(R.1) $E(x, x) = 1$
(R.2) $E(x, y) = E(y, x)$
(R.3) $d(x, y) \leq d(z, u)$ *implies* $E(x, y) \geq E(z, u)$

[2]For a more general approach, see [4].

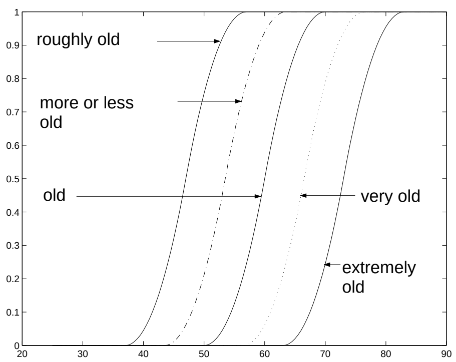

Figure 2: old and modified terms

(R.3) is an immediate translation of Assumption 1. In [5, 6] it is shown that the proposed framework for representing linguistic modifiers works well in numerical as well as non-numerical universes, for all kinds of membership functions, provided that resemblance relations are used to model approximate equality. Figure 2 shows a fuzzy set A representing old. The universe of ages is equipped with the pseudo-metric $d_{|\cdot|}(x, y) = |x - y|$. Using the resemblance relation E_1 defined by $E_1(x, y) = \min(1, \max(0, 3 - 0.3|x - y|, 0))$, membership functions for very old ($E_1^{\heartsuit}(A)$) and for more or less old ($E_1^{\clubsuit}(A)$) are generated. In the same example the fuzzy sets $E_1^{\heartsuit}(E_1^{\heartsuit}(A))$ and $E_1^{\clubsuit}(E_1^{\clubsuit}(A))$ are used to represent extremely old and roughly old respectively — they are clearly different from very old and more or less old. In all cases $\mathcal{T}_L$ and $\mathcal{I}_{\mathcal{T}_L}$ are used.

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