Fuzzy Topologies Induced by Fuzzy Relation Based Modifiers

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Abstract. In this paper we highlight the specific meaning of images of fuzzy sets under fuzzy relations in the context of fuzzy topology. More precisely we show that fuzzy modifiers taking direct and superdirect images of fuzzy sets under fuzzy pre-orderings are respectively closure and interior operators, inducing fuzzy topologies. Furthermore we investigate under which conditions the same applies to the recently introduced general closure and opening operators based on arbitrary fuzzy relations.

1 Introduction

Images of fuzzy sets under fuzzy relations prove to be very powerful tools in a wide range of applications, varying from fuzzy databases, over fuzzy morphology, fuzzy rough set theory, and the representation of linguistic modifiers, to approximate reasoning schemes [5]. In this paper we will show that they have a specific meaning in the context of fuzzy topology as well. More specifically in Section 2 we will show that fuzzy modifiers taking direct and superdirect images of fuzzy sets under fuzzy pre-orderings are respectively closure and interior operators inducing Chang fuzzy topologies. In Section 3 we will investigate under which conditions the same holds for the general closure and opening operators introduced recently by Bodenhofer [2].

Throughout the paper T will denote a triangular norm with left-continuous partial mappings $T(x, .)$ for all x in $[0, 1]$. Furthermore $\vec{T}$ will denote its residual implication defined by

$$\vec{T}(x, y) = \sup\{\lambda \in [0, 1] | T(\lambda, x) \leq y\}$$

It can be verified then that $\vec{T}$ is non-increasing and left-continuous in the first argument while non-decreasing and right-continuous in the second. Furthermore for all x, y, and z in $[0, 1]$, and $\{y_i \mid i \in I\}$ a family in $[0, 1]$:

(1) $T(0, x) = 0$
(2) $T(x, \sup\limits_{i \in I} y_i) = \sup\limits_{i \in I} T(x, y_i)$
(3) $\vec{T}(x, y) = 1$ iff $x \leq y$

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(4) $\vec{T}(1, x) = x$
(5) $\vec{T}(x, \inf_{i \in I} y_i) = \inf_{i \in I} \vec{T}(x, y_i)$
(6) $\vec{T}(T(x, y), z) \leq \vec{T}(x, \vec{T}(y, z))$

Let X be a non-empty set. The class of all fuzzy sets on X will be denoted $\mathcal{F}(X)$. As usual union, intersection and complement of fuzzy sets are defined by, for x in X,

$$\bigcup_{i \in I} A_i(x) = \sup_{i \in I} A_i(x)$$

$$\bigcap_{i \in I} A_i(x) = \inf_{i \in I} A_i(x)$$

$$(co\ A)(x) = 1 - A(x)$$

in which $\{A_i | i \in I\}$ is a family of fuzzy sets on X, and A in $\mathcal{F}(X)$. Inclusion for two fuzzy sets A and B on X is defined as:

$$A \subseteq B \text{ iff } A(x) \leq B(x) \text{ for all } x \text{ in } X$$

Definition 1 (Fuzzy modifier). *[16] A fuzzy modifier on X is an $\mathcal{F}(X)$-$\mathcal{F}(X)$ mapping.*

In [4] the following class of fuzzy modifiers based on fuzzy relations is defined:

Definition 2 (Fuzzy relation based modifiers). *Let R be a fuzzy relation on X, i.e. $R \in \mathcal{F}(X \times X)$. The fuzzy modifiers $R\!\uparrow$ and $R\!\downarrow$ on X are defined by, for A in $\mathcal{F}(X)$ and y in X:*

$$R\!\uparrow\! A(y) = \sup_{x \in X} T(A(x), R(x, y))$$

$$R\!\downarrow\! A(y) = \inf_{x \in X} \vec{T}(R(y, x), A(x))$$

$R\!\uparrow\! A$ is also called the direct image of A under R, while $R\!\downarrow\! A$ is the superdirect image of A under R^{-1} (i.e. the inverse relation of R) [12]. In fuzzy set theoretical settings, fuzzy modifiers are usually associated with the representation of linguistic hedges such as very, more or less, rather,... (see e.g. [7], [13], [16]). The class of fuzzy modifiers based on fuzzy relations as defined above however can be applied to a wide range of other purposes as well. In [4], [10] it is shown that for a suitable fuzzy relation R they correspond to the dilation and erosion operators of fuzzy morphology (used for image processing), while in [6], [10] it is illustrated that they can be used as fuzzy-rough approximators (for dealing with incomplete information). In this paper we will show that for R a fuzzy pre-ordering the R-based fuzzy modifiers defined in Definition 2 also are closure and interior operators inducing fuzzy topologies (in the sense of Chang [3]).

Definition 3 (Fuzzy T-preordering). *A fuzzy relation R on X is called a fuzzy T-preordering w.r.t. a t-norm T iff for all x, y and z in X:*