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OPENINGS AND CLOSURES OF FUZZY PREORDERINGS: THEORETICAL BASICS AND APPLICATIONS TO FUZZY RULE-BASED SYSTEMS

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The purpose of this paper is two-fold. Firstly, a general concept of closedness of fuzzy sets under fuzzy preorderings is proposed and investigated along with the corresponding opening and closure operators. Secondly, the practical impact of this notion is demonstrated by applying it to the analysis of ordering-based modifiers.

Keywords: Closedness; Fuzzy preordering; Fuzzy relation; Linguistic modifier

1. INTRODUCTION

Images of fuzzy sets under fuzzy relations have been investigated mainly in two contexts: on the one hand, mostly under the term "full image" (Gottwald, 1993), they can be regarded as very general tools for fuzzy inference, leading to the so-called "compositional rule of inference" (Gottwald, 1993; Bauer *et al.*, 1995). The theory of fuzzy relational equations makes direct use of this fundamental principle, too (Sanchez, 1984; Miyakoshi and Shimbo, 1985; di Nola *et al.*, 1991; Gottwald, 1993; De Baets, 2000). On the other hand, under the term "extensional hull", the image of a fuzzy set under a fuzzy equivalence relation yields the smallest fuzzy superset which is "closed" under the relation. This closedness property is usually called "extensionality" (Kruse *et al.*, 1994). The concepts of extensionality and extensional hulls have turned out to be extremely useful, in particular when the analysis and interpretation of fuzzy partitions and controllers is concerned (Klawonn, 1993; Klawonn and Kruse, 1993; Klawonn and Castro, 1995; Klawonn *et al.*, 1995).

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In the first part of this paper, we would like to generalize the concept of extensionality to arbitrary reflexive and T-transitive fuzzy relations—so-called fuzzy preorderings. Based on this general and powerful notion, smallest closed supersets and largest closed fuzzy subsets will be studied. It will turn out that again the two very common concepts of images under fuzzy relations are obtained.

The second part is devoted to a new view on these images of fuzzy sets under fuzzy relations—making use of the results on closedness and the corresponding closure operator, we are able to provide a new framework for defining the ordering-based modifiers "at least" and "at most".

2. PRELIMINARIES

Throughout the whole paper, we will not explicitly distinguish between fuzzy sets and their corresponding membership functions. Consequently, uppercase letters will be used for both synonymously. The set of all fuzzy sets on a domain X will be denoted with $\mathscr{F}(X)$.

For intersecting and unifying fuzzy sets, we will suffice with minimum and maximum:

$$(A \cap B)(x) = \min(A(x), B(x))$$
$$(A \cup B)(x) = \max(A(x), B(x)).$$

In general, aside from intersections and unions of fuzzy sets, triangular norms (Klement *et al.*, 2000) will be considered as our standard models of conjunction.

Definition 1 A *triangular norm* (*t-norm* for short) is an associative, commutative, and non-decreasing binary operation on the unit interval (i.e. a $[0,1]^2 \to [0,1]$ mapping) which has 1 as neutral element.

In this paper, unless stated otherwise, assume that T denotes a left-continuous triangular norm, i.e. a t-norm whose partial mappings $T(x,.)$ and $T(.,x)$ are left-continuous.

Correspondingly, so-called residual implications are used as the concepts of logical implication. In order to provide the reader with the basic properties of residual implications, we will now briefly recall them. For proofs, the reader is referred to the literature (Gottwald, 1993; Hájek, 1998).

Definition 2 A mapping $R : [0,1]^2 \to [0,1]$ is called *residual implication (residuum)* of T if and only if the following equivalence is fulfilled for all $x, y, z \in [0,1]$:

$$T(x, y) \leq z \Leftrightarrow x \leq R(y, z). \tag{1}$$

Lemma 3 For any left-continuous t-norm T, there exists a unique residuum $\vec{T}$ given as

$$\vec{T}(x, y) = \sup\{u \in [0,1] | T(u, x) \leq y\}.$$

Only briefly, we mention the concept of logical equivalence induced by a left-continuous t-norm.

Definition 4 The *biimplication* $\overleftrightarrow{T}$ of T is defined as

$$\overleftrightarrow{T}(x, y) = \min(\vec{T}(x, y), \vec{T}(y, x)).$$

For elementary properties of the fuzzy logical operations $\vec{T}$ and $\overleftrightarrow{T}$, the reader is referred to the relevant literature (Gottwald, 1993; Fodor and Roubens, 1994; Hájek, 1998; Klement *et al.*, 2000; Gottwald, 2001).