Fuzzy Rough Sets: Beyond the Obvious

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***Abstract*— Rough set theory was introduced in 1982. Soon it was combined with fuzzy set theory, giving rise to a hybrid model, involving fuzzy sets and fuzzy relations, which appears to be a natural, elegant generalization. In this paper we reveal that in the fuzzification process an important step seems to be overlooked. The most fascinating part is that this forgotten step arises from the true essence of fuzzy set theory: namely, that an element can belong to a given degree to more than one fuzzy set at the same time.**

I. Introduction

Pawlak [12] launched rough set theory as a framework for the construction of approximations of concepts when only incomplete information is available. The available information consists of a set A of examples (a subset of a universe X, X being a non-empty set of objects we want to say something about) of a concept C, and a relation R in X. R models "indiscernibility" or "indistinguishability" and therefore generally is a tolerance relation (i.e. a reflexive and symmetrical relation) and in most cases even an equivalence relation (i.e. a transitive tolerance relation). Rough set analysis makes statements about the membership of some element y of X to the concept C of which A is a set of examples, based on the indistinguishability between y and the elements of A. To arrive at such statements, A is approximated in two ways. An element y of X belongs to the lower approximation of A if the equivalence class to which y belongs is included in A. On the other hand y belongs to the upper approximation of A if its equivalence class has a non–empty intersection with A.

After a public debate reflecting rivalry between this new theory and the slightly older fuzzy set theory, many people have worked on the fuzzification of upper– and lower approximations (e.g. [6], [11], [14], [15], [17]). In doing so, the central focus moved from elements' indistinguishability (w.r.t. their attribute values in an information system) to their similarity—represented by a fuzzy relation R—again w.r.t. to those attribute values: objects are categorized into classes with "soft" boundaries based on their similarity to one another. A concrete advantage of such a scheme is that abrupt transitions between classes are replaced by gradual ones, allowing that an element can belong (to varying degrees) to more than one class. On another count the set A to be approximated can be fuzzy as well in the new hybrid model, which is called "fuzzy rough set theory".

The most striking aspect of all the studies mentioned above is that none of them tries to exploit the fact that an element y of X can belong to some degree to several "soft similarity classes" at the same time. This property does not only lie at the heart of fuzzy set theory but is also crucial in the decision on how to define lower and upper approximations. For instance, as mentioned above, in traditional rough set theory, y belongs to the lower approximation of A if the equivalence class to which y belongs is included in A. But what happens if y belongs to several "fuzzy equivalence classes" at the same time? Do we then require that all of them are included in A? Most of them? Or just one? And then, which one?

Traditional fuzzy rough set theory involves only one fuzzy equivalence class. In this paper we explore what happens if we abandon this most obvious choice. After recalling the necessary preliminaries in Section 2, in Section 3 we define alternative lower and upper approximations of a (fuzzy) set A, based on different choices about which fuzzy equivalence classes should be included in, or have a non-empty intersection, with A. In Section 4 we examine their properties, paying significant attention to the role that the $\mathcal{T}$-transitivity of the fuzzy relation R plays in this game. This allows us to end with an interesting conclusion and some ideas for further research.

II. Preliminaries

Throughout this paper, let $\mathcal{T}$ and $\mathcal{I}$ denote a triangular norm and an implicator respectively. Recall that a triangular norm (t–norm for short) $\mathcal{T}$ is any increasing, commutative and associative $[0,1]^2 \to [0,1]$ mapping satisfying $\mathcal{T}(1,x) = x$, for all x in $[0,1]$. A negator $\mathcal{N}$ is a decreasing $[0,1] \to [0,1]$ mapping satisfying $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$. $\mathcal{N}$ is called involutive if $\mathcal{N}(\mathcal{N}(x)) = x$ for all x in $[0,1]$. Finally, an implicator is any $[0,1]^2 \to [0,1]$–mapping $\mathcal{I}$ satisfying $\mathcal{I}(0,0) = 1, \mathcal{I}(1,x) = x$, for all x in $[0,1]$. Moreover we require $\mathcal{I}$ to be decreasing in its first, and increasing in its second component. If $\mathcal{T}$ is a t–norm, the mapping $\mathcal{I}_{\mathcal{T}}$ defined by, for all x and y in [0,1],

$$\mathcal{I}_{\mathcal{T}}(x,y) = \sup\{\lambda | \lambda \in [0,1] \text{ and } \mathcal{T}(x,\lambda) \leq y\} \qquad (1)$$

is an implicator, usually called the residual implicator (of $\mathcal{T}$). If $\mathcal{T}$ is a t–norm and $\mathcal{N}$ is an involutive negator, then the mapping $\mathcal{I}_{\mathcal{T},\mathcal{N}}$ defined by, for all x and y in [0,1],

$$\mathcal{I}_{\mathcal{T},\mathcal{N}}(x,y) = \mathcal{N}(\mathcal{T}(x,\mathcal{N}(y))) \qquad (2)$$