On the Construction of Interval-Valued Fuzzy Morphological Operators

Tom Mélange[a], Mike Nachtegael[a], Peter Sussner[b], Etienne E. Kerre[a]

[a]Fuzziness and Uncertainty Modelling Research Unit, Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281 (Building S9), 9000 Ghent, Belgium
[b]Department of Applied Mathematics, University of Campinas, Campinas, SP 13083 859, Brazil

Abstract

Classical fuzzy mathematical morphology is one of the extensions of original binary morphology to greyscale morphology. Recently, this theory was further extended to interval-valued fuzzy mathematical morphology by allowing uncertainty in the grey values of the image and the structuring element. In this paper, we investigate the construction of increasing interval-valued fuzzy operators from their binary counterparts and work this out in more detail for the morphological operators, which results in a nice theoretical link between binary and interval-valued fuzzy mathematical morphology. The investigation is done both in the general continuous and the practical discrete case. It will be seen that the characterization of the supremum in the discrete case leads to stronger relationships than in the continuous case.

Key words: image processing, mathematical morphology, interval-valued

Email addresses: `tom.melange@ugent.be` (Tom Mélange), `mike.nachtegael@ugent.be` (Mike Nachtegael), `sussner@ime.unicamp.br` (Peter Sussner), `etienne.kerre@ugent.be` (Etienne E. Kerre)
URL: `http://www.fuzzy.ugent.be` (Etienne E. Kerre)

fuzzy sets, α-cuts

1. Introduction

The image processing domain consists of numerous theories to extract information (edges, patterns, ...) from images. Mathematical morphology is among these theories. In this theory [3, 1, 2], images are elements of a complete lattice (L, C) with L the space and C the comparison operator. Morphological operators have certain properties with respect to the comparison operator. For instance, dilations commute with supremum. Many morphological operators transform images with the help of subsets called structuring elements. Originally, only binary (black and white) images and structuring elements were considered. Next, binary morphology [4], was extended to greyscale images in three different ways. In the *threshold approach* [4], the structuring element still had to be binary, while in the *umbra approach* [5], also greyscale structuring elements were allowed. A third approach, *fuzzy mathematical morphology* [6], was introduced some time later and was inspired by the observation that both greyscale images and fuzzy sets are modelled as mappings from a universe $\mathcal{U}$ into the unit interval $[0, 1]$. Fuzzy set theory is thus used as a tool here and not to model uncertainty. Besides the extensions from greyscale morphology to multivalued morphology (e.g. [7, 8]), recently, greyscale fuzzy mathematical morphology was also further extended based on extensions of fuzzy sets to be able to deal with uncertain and bipolar information [9, 10, 11, 12, 13]. The interval-valued fuzzy extension introduced in [12, 13], on which we concentrate in this paper, now has the important feature that it is no longer only used as a tool, but also as a model to represent uncertainty regarding the measured grey levels. In this model, a pixel in the image domain is no longer mapped onto one specific grey value ($\in [0, 1]$), but onto an interval of grey values to which the uncertain grey value is expected to belong. For a discussion on the interval-valued image model, we refer to [12, 19].

As a side note, we mention here that interval-valued fuzzy set theory and intuitionistic fuzzy set theory [14] are equivalent [15] and as a consequence also interval-valued and intuitionistic fuzzy mathematical morphology. So the results in this paper can be translated to the intuitionistic/bipolar model [9, 10] straightforwardly.

Finally, we remark that interval-valued representations also occur in a natural way in other image processing subdomains. Examples can be found in inverse halftoning [16], in the context of wavelets [17] and in edge detection applications [18]. In the latter example, the interval-valued representation is however rather a tool than a model.

In our previous work [19], we already studied the decomposition of the interval-valued fuzzy morphological operators into their cuts. More precisely, we investigated the relationships between the cuts of an interval-valued fuzzy morphological operator and the corresponding binary operator applied on the cuts of the image and structuring element. In this paper, we tackle the reverse problem, i.e., the construction of interval-valued fuzzy morphological operators from the binary operators. We start from a more general perspective, in which we investigate the construction of increasing interval-valued fuzzy operators from binary ones and additionally apply the construction principle on the binary dilation that is increasing w.r.t. to both the image and structuring element.

This is first of all interesting from a theoretical point of view since it provides us a link between binary and interval-valued fuzzy mathematical morphology. It allows us to compute the interval-valued operators by combining the results of several binary operators and also to approximate them by a finite number of binary operators. We perform this investigation both in the general continuous framework and the practical discrete framework. In practice, image domains and allowed grey values have namely been sampled down due to technical limitations. As will be shown, some stronger relationships hold in this discrete case.

The paper is organized as follows. We repeat the basic principles of interval-valued fuzzy mathematical morphology in section 2. Section 3 then studies the construction of the interval-valued fuzzy morphological operators based on weak and strict $[\alpha_1, \alpha_2]$-cuts in a continuous framework. Section 4 deals with the construction in a discrete framework, which leads to slightly different results. Section 5 concludes the paper.

2. Interval-Valued Fuzzy Mathematical Morphology

An interval-valued fuzzy set [20] is an extension of a classical fuzzy set [21]. Whereas a fuzzy set F in a universe $\mathcal{U}$ maps every element $u \in \mathcal{U}$ onto its membership degree $F(u) \in [0,1]$ in the set F, an interval-valued fuzzy set G in the universe $\mathcal{U}$ maps every $u \in \mathcal{U}$ onto a closed subinterval $G(u) = [G_1(u), G_2(u)]$ of $[0,1]$, in this way allowing uncertainty about the membership degree. An interval-valued fuzzy set in a universe $\mathcal{U}$ is thus modelled as an $\mathcal{U}$-L^I mapping, with $L^I = \{[x_1, x_2] \mid [x_1, x_2] \subseteq [0,1]\}$. The lower and upper bound of an element x of L^I will be denoted by x_1 and x_2 respectively: $x = [x_1, x_2]$ (Fig 1). We will further use the notation $\mathcal{IVFS}(\mathcal{U})$ for the class of interval-valued fuzzy sets over the universe $\mathcal{U}$.

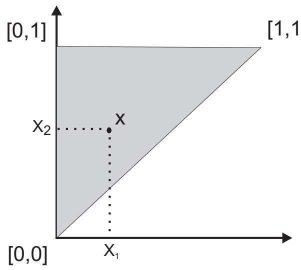

Figure 1: Graphical representation of L^I.

In [15] it is shown that for the partial ordering $\leq_{L^I}$ on L^I given by

$$x \leq_{L^I} y \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2,\ \forall x, y \in L^I,$$

the structure $(L^I, \leq_{L^I})$ forms a complete lattice (which is a necessary and sufficient condition to do morphology on L^I). The infimum and supremum of an arbitrary subset S of L^I are then respectively given by $\inf S = [\inf\limits_{x \in S} x_1, \inf\limits_{x \in S} x_2]$ and $\sup S = [\sup\limits_{x \in S} x_1, \sup\limits_{x \in S} x_2]$. For $\inf L^I = [0, 0]$ and $\sup L^I = [1, 1]$ we will use the notations 0_{L^I} and 1_{L^I} respectively. Further, the union of an arbitrary family $(A_j)_{j \in J}$ of interval-valued fuzzy sets on $\mathcal{U}$ is defined by $(\bigcup\limits_{j \in J} A_j)(u) = \sup\limits_{j \in J} A_j(u), \forall u \in \mathcal{U}$.

Other related orderings on L^I that will occur in the remainder of the paper are ($\forall x, y \in L^I$):

$$\begin{aligned} x <_{L^I} y &\Leftrightarrow x \leq_{L^I} y \text{ and } x \neq y, \\ x \ll_{L^I} y &\Leftrightarrow x_1 < y_1 \text{ and } x_2 < y_2, \\ x \geq_{L^I} y &\Leftrightarrow y \leq_{L^I} x, \\ x >_{L^I} y &\Leftrightarrow y <_{L^I} x, \\ x \gg_{L^I} y &\Leftrightarrow y \ll_{L^I} x. \end{aligned}$$

Remark that a partial ordering on L^I also induces a partial ordering on $\mathcal{IVFS}(\mathcal{U})$.

Since we investigate the construction of the interval-valued fuzzy dilation, erosion, opening and closing from the corresponding binary operators, we first review the definition of the basic binary morphological operators.

Definition 1. [4] *Let $A, B \subseteq \mathbb{R}^n$. The binary dilation $D(A, B)$, the binary erosion $E(A, B)$, the binary closing $C(A, B)$ and the binary opening $O(A, B)$ are the sets given by:*

$$\begin{aligned} D(A, B) &= \{y \mid T_y(-B) \cap A \neq \emptyset\}, \\ E(A, B) &= \{y \mid T_y(B) \subseteq A\}, \\ C(A, B) &= E(D(A, B), B), \\ O(A, B) &= D(E(A, B), B), \end{aligned}$$

with $T_y(B) = \{x \in \mathbb{R}^n \mid x - y \in B\}$ and $-B = \{-b \mid b \in B\}$.

As can be seen, the intersection and inclusion of two sets plays an important role in the definition of the morphological operators. Therefore, the underlying Boolean conjunction ($\wedge$: *and*)(an element belongs to the first and to the second set) and Boolean implication ($\Rightarrow$: *implies*)(if an element belongs to the first set, then it also belongs to the second set) need to be extended to the interval-valued case if we want to introduce the interval-valued fuzzy morphological operators. Also an extension of the Boolean negation ($\neg$: *not*) will be needed in the paper.

Definition 2. [22] *A negator $\mathcal{N}$ on L^I is a decreasing $L^I - L^I$ mapping that coincides with the Boolean negation on $\{0,1\}$ ($\mathcal{N}(0_{L^I}) = 1_{L^I}$ and $\mathcal{N}(1_{L^I}) = 0_{L^I}$). It is called an involutive negator on L^I if $(\forall x \in L^I)(\mathcal{N}(\mathcal{N}(x)) = x)$.*

An example of an involutive negator on L^I is the standard negator $\mathcal{N}_s$, defined by $\mathcal{N}_s([x_1,x_2]) = [1-x_2, 1-x_1]$, for all $x = [x_1,x_2] \in L^I$.

Definition 3. [22] *A conjunctor $\mathcal{C}$ on L^I is an increasing $(L^I)^2 - L^I$ mapping that coincides with the Boolean conjunction on $\{0,1\}^2$ (i.e., $\mathcal{C}(0_{L^I}, 0_{L^I}) = \mathcal{C}(0_{L^I}, 1_{L^I}) = \mathcal{C}(1_{L^I}, 0_{L^I}) = 0_{L^I}$ and $\mathcal{C}(1_{L^I}, 1_{L^I}) = 1_{L^I}$). If it also satisfies $(\forall x \in L^I)(\mathcal{C}(1_{L^I}, x) = \mathcal{C}(x, 1_{L^I}) = x)$ then it is called a semi-norm on L^I. If it is further also commutative and associative, we call it a t-norm on L^I.*

An example of a t-norm on L^I is the conjunctor $\mathcal{C}_{\min}$, defined by $\mathcal{C}_{\min}(x,y) = [\min(x_1,y_1), \min(x_2,y_2)]$, for all $(x,y) \in (L^I)^2$.

Definition 4. [22] *An implicator $\mathcal{I}$ on L^I is a hybrid monotonic $(L^I)^2 - L^I$ mapping (i.e., decreasing in the first and increasing in the second argument) that coincides with the Boolean implication on $\{0,1\}^2$ (i.e., $\mathcal{I}(0_{L^I}, 0_{L^I}) = \mathcal{I}(0_{L^I}, 1_{L^I}) = \mathcal{I}(1_{L^I}, 1_{L^I}) = 1_{L^I}$ and $\mathcal{I}(1_{L^I}, 0_{L^I}) = 0_{L^I}$). If it satisfies $(\forall x \in L^I)(\mathcal{I}(1_{L^I}, x) = x)$ then it is called a border implicator on L^I. If it is contrapositive w.r.t. its induced negator $\mathcal{N}_{\mathcal{I}}$ ($\mathcal{N}_{\mathcal{I}}(x) = \mathcal{I}(x, 0_{L^I}), \forall x \in L^I$), i.e.*

$(\forall(x,y) \in (L^I)^2)(\mathcal{I}(x,y) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}(y), \mathcal{N}_{\mathcal{I}}(x)))$, *and if it fulfils the exchange principle, i.e.,* $(\forall(x,y,z) \in (L^I)^3)(\mathcal{I}(x,\mathcal{I}(y,z)) = \mathcal{I}(y,\mathcal{I}(x,z)))$, *we call it a model implicator on* L^I.

An example of a model implicator on L^I is the implicator $\mathcal{I}_{\min,\mathcal{N}_s}$ which is given by $\mathcal{I}_{\min,\mathcal{N}_s}(x,y) = [\max(1-x_2,y_1), \max(1-x_1,y_2)]$, for all $(x,y) \in (L^I)^2$.

With the support d_A of the interval-valued fuzzy set A in $\mathbb{R}^n$ and the reflection $-B$ of the interval-valued fuzzy set B in $\mathbb{R}^n$ respectively defined by $d_A = \{x \mid x \in \mathbb{R}^n \text{ and } A(x) \neq 0_{L^I}\}$ and $(-B)(x) = B(-x), \forall x \in \mathbb{R}^n$, we can now give the definitions of the interval-valued fuzzy morphological operators.

Definition 5. [*10, 11*] *Let* $\mathcal{C}$ *be a conjunctor on* L^I, *let* $\mathcal{I}$ *be an implicator on* L^I, *and let* $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$. *The interval-valued fuzzy dilation* $D^I_{\mathcal{C}}(A,B)$ *is the interval-valued fuzzy set in* $\mathbb{R}^n$ *defined by:*

$$D^I_{\mathcal{C}}(A,B)(y) = \sup_{x \in T_y(-d_B) \cap d_A} \mathcal{C}(B(y-x), A(x)), \forall y \in \mathbb{R}^n. \tag{1}$$

Remark that if $y \notin D(d_A, d_B)$, *then* $D^I_{\mathcal{C}}(A,B)(y) = 0_{L^I}$.

The interval-valued fuzzy erosion $E^I_{\mathcal{I}}(A,B)$ *is the interval-valued fuzzy set in* $\mathbb{R}^n$ *defined by:*

$$E^I_{\mathcal{I}}(A,B)(y) = \inf_{x \in T_y(d_B)} \mathcal{I}(B(x-y), A(x)), \forall y \in \mathbb{R}^n. \tag{2}$$

The interval-valued fuzzy closing $C^I_{\mathcal{C},\mathcal{I}}(A,B)$ *and the interval-valued fuzzy opening* $O^I_{\mathcal{C},\mathcal{I}}(A,B)$ *are the interval-valued fuzzy sets in* $\mathbb{R}^n$ *defined by:*

$$C^I_{\mathcal{C},\mathcal{I}}(A,B) = E^I_{\mathcal{I}}(D^I_{\mathcal{C}}(A,B), B), \tag{3}$$

$$O^I_{\mathcal{C},\mathcal{I}}(A,B) = D^I_{\mathcal{C}}(E^I_{\mathcal{I}}(A,B), B). \tag{4}$$

For a list of the basic properties of these operators, we refer the reader to [24]. An illustration of the interval-valued morphological operators can for example be found in [12, 19]

3. Construction of Interval-valued Fuzzy Morphological Operators - Continuous Case

In this section, we investigate the construction of interval-valued fuzzy morphological operators based on weak and strict $[\alpha_1, \alpha_2]$-cuts in a continuous framework. We start from a more general perspective and investigate the construction of an interval-valued fuzzy set from an increasing family of crisp sets in Subsection 3.2.1. In Subsection 3.2.2 these results are then used to extend increasing operators defined on crisp sets to operators on interval-valued fuzzy sets. In particular, the obtained construction principle is applied to the binary dilation that is increasing w.r.t. to both the image and structuring element. First, we give the definitions of the different $[\alpha_1, \alpha_2]$-cuts of an interval-valued fuzzy set.

3.1. The Different $[\alpha_1, \alpha_2]$-cuts

The different $[\alpha_1, \alpha_2]$-cuts are defined as follows [23] (with $U_{L^I} = \{[x_1, x_2] \in L^I \mid x_2 = 1\}$):

Definition 6. *Let $A \in \mathcal{IVFS}(\mathbb{R}^n)$. For $[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}$, the weak $[\alpha_1, \alpha_2]$-cut $A_{\alpha_1}^{\alpha_2}$ of A is defined as:*

$$\begin{aligned} A_{\alpha_1}^{\alpha_2} &= \{x \mid x \in \mathbb{R}^n,\ A_1(x) \geq \alpha_1 \text{ and } A_2(x) \geq \alpha_2\} \\ &= \{x \mid x \in \mathbb{R}^n \text{ and } A(x) \geq_{L^I} [\alpha_1, \alpha_2]\}. \end{aligned}$$

For $[\alpha_1, \alpha_2] \in L^I \backslash U_{L^I}$*, the strict* $[\alpha_1, \alpha_2]$*-cut* $A_{\overline{\alpha_1}}^{\overline{\alpha_2}}$ *of* A *is defined as:*

$$\begin{aligned} A_{\overline{\alpha_1}}^{\overline{\alpha_2}} &= \{x \mid x \in \mathbb{R}^n,\ A_1(x) > \alpha_1 \textit{ and } A_2(x) > \alpha_2\} \\ &= \{x \mid x \in \mathbb{R}^n \textit{ and } A(x) \gg_{L^I} [\alpha_1, \alpha_2]\}. \end{aligned}$$

Remark that since $\{x \mid x \in \mathbb{R}^n,\ A_1(x) \geq 0 \text{ and } A_2(x) \geq 0\} = \mathbb{R}^n$ and since $\{x \mid x \in \mathbb{R}^n \text{ and } A_2(x) > 1\} = \emptyset$, these cases do not yield new information and consequently $[\alpha_1, \alpha_2] = 0_{L^I}$ and $[\alpha_1, \alpha_2] \in U_{L^I}$ are excluded for respectively the weak and the strict $[\alpha_1, \alpha_2]$-cut.

3.2. Construction based on weak $[\alpha_1, \alpha_2]$*-cuts*

3.2.1. Introduction

Definition 7. [23] *The product of a crisp set* $C \subset \mathbb{R}^n$ *and an interval* $[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}$ *is defined as the interval-valued fuzzy set* $[\alpha_1, \alpha_2]C$*, given by:*

$$[\alpha_1, \alpha_2]C(x) = \begin{cases} [\alpha_1, \alpha_2] & \textit{if } x \in C \\ 0_{L^I} & \textit{else} \end{cases},\ \forall x \in \mathbb{R}^n. \tag{5}$$

By using the interval-valued fuzzy sets $[\alpha_1, \alpha_2]A_{\alpha_1}^{\alpha_2}$, based on the weak cuts of an interval-valued fuzzy set $A \in \mathcal{IVFS}(\mathbb{R}^n)$, the original interval-valued fuzzy set A can be reconstructed as follows.

Lemma 1. [23] *Let* $A \in \mathcal{IVFS}(\mathbb{R}^n)$*. It holds that* $A = \bigcup\limits_{[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}} [\alpha_1, \alpha_2]A_{\alpha_1}^{\alpha_2}$*, i.e.:*

$$A(x) = \sup_{[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}} ([\alpha_1, \alpha_2]A_{\alpha_1}^{\alpha_2})(x) = \sup\{[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\} \mid x \in A_{\alpha_1}^{\alpha_2}\},\ \forall x \in \mathbb{R}^n.$$

If we now consider a family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\}}$ of crisp subsets of $\mathbb{R}^n$ that is decreasing ($[\alpha_1,\alpha_2] \leq_{L^I} [\alpha_3,\alpha_4] \Rightarrow P_{[\alpha_1,\alpha_2]} \supseteq P_{[\alpha_3,\alpha_4]}$) and we define the interval-valued fuzzy set R in $\mathbb{R}^n$ for all $x \in \mathbb{R}^n$ as,

$$\begin{aligned} R(x) &= \sup_{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\}} ([\alpha_1,\alpha_2]P_{[\alpha_1,\alpha_2]})(x) \qquad (6)\\ &= \sup\{[\alpha_1,\alpha_2] \in L^I\setminus\{0_{L^I}\} \mid x \in P_{[\alpha_1,\alpha_2]}\}, \end{aligned}$$

then we might wonder whether it holds that $(\forall[\alpha_1,\alpha_2] \in L^I\setminus\{0_{L^I}\})(R_{\alpha_1}^{\alpha_2} = P_{[\alpha_1,\alpha_2]})$. In any case, the following inclusion always holds:

Proposition 1. *For a decreasing family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\}}$ of crisp subsets of $\mathbb{R}^n$ and the interval-valued fuzzy set R defined in (6), it holds that:*

$$(\forall[\alpha_1,\alpha_2] \in L^I\setminus\{0_{L^I}\})(P_{[\alpha_1,\alpha_2]} \subseteq R_{\alpha_1}^{\alpha_2}).$$

Proof. Let $[\beta_1,\beta_2] \in L^I\setminus\{0_{L^I}\}$ and let $x \in P_{[\beta_1,\beta_2]}$. It then holds that:

$$\begin{aligned} x \in P_{[\beta_1,\beta_2]} &\Leftrightarrow [\beta_1,\beta_2] \in \{[\alpha_1,\alpha_2] \in L^I\setminus\{0_{L^I}\} \mid x \in P_{[\alpha_1,\alpha_2]}\}\\ &\Rightarrow \sup\{[\alpha_1,\alpha_2] \in L^I\setminus\{0_{L^I}\} \mid x \in P_{[\alpha_1,\alpha_2]}\} \geq_{L^I} [\beta_1,\beta_2]\\ &\Leftrightarrow R(x) \geq_{L^I} [\beta_1,\beta_2]\\ &\Leftrightarrow x \in R_{\beta_1}^{\beta_2}. \end{aligned}$$

As a consequence, $P_{[\beta_1,\beta_2]} \subseteq R_{\beta_1}^{\beta_2}$. □

However, we do not neccessarily have an equality.

Example 1. Let $P_{[\alpha_1,\alpha_2]} =]-1+\alpha_1, 1-\alpha_2[$ for all $[\alpha_1,\alpha_2] \in L^I\setminus\{0_{L^I}\}$. For $[\beta_1,\beta_2] \ll_{L^I} [\alpha_1,\alpha_2]$ we have that $-1+\beta_1 < -1+\alpha_1 \leq 1-\alpha_2 < 1-\beta_2$ or thus $-1+\alpha_1 \in P_{[\beta_1,\beta_2]}$ and $1-\alpha_2 \in P_{[\beta_1,\beta_2]}$. As a consequence $R(-1+\alpha_1) = \sup\{[\beta_1,\beta_2] \in L^I\setminus\{0_{L^I}\} \mid -1+\alpha_1 \in P_{[\beta_1,\beta_2]}\} = [\alpha_1,\alpha_2]$ and analoguously

$R(1-\alpha_2) = [\alpha_1, \alpha_2]$, thus $-1+\alpha_1 \in R_{\alpha_1}^{\alpha_2}$ and $1-\alpha_2 \in R_{\alpha_1}^{\alpha_2}$, what means that $R_{\alpha_1}^{\alpha_2} \neq P_{[\alpha_1,\alpha_2]}$.

$\diamondsuit$

The reverse inclusion (and thus the equality) only holds under certain conditions. To formulate these conditions, we define the set d_P as

$$d_P = \{x \in \mathbb{R}^n \mid (\exists[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\})(x \in P_{[\alpha_1,\alpha_2]})\}. \tag{7}$$

Further, for a fixed point $x \in d_P$, we introduce the set S_x, given by

$$S_x = \{[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\} \mid x \in P_{[\alpha_1,\alpha_2]}\}, \tag{8}$$

and we denote the supremum of this set by $s_x = [s_{x,1}, s_{x,2}]$:

$$s_x = \sup S_x. \tag{9}$$

Remark that $S_x \neq \emptyset$.

The conditions under which the equality holds, are given in the following Proposition:

Proposition 2. *For a decreasing family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I\backslash\{0_{L^I}\}}$ of crisp subsets of $\mathbb{R}^n$, the interval-valued fuzzy set R defined in (6) and the sets d_P and S_x and the supremum s_x of the latter set, respectively defined in expressions (7)-(9), it holds that:*

$$(\forall[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\})(P_{[\alpha_1,\alpha_2]} = R_{\alpha_1}^{\alpha_2}) \Leftrightarrow (\forall x \in d_P)(s_x \in S_x).$$

Proof.

$\Leftarrow$: Follows from the proof of Proposition 1. Since $s_x(= \sup S_x) \in S_x$ it now also holds that $\sup S_x \geq_{L^I} [\beta_1, \beta_2] \Rightarrow [\beta_1, \beta_2] \in S_x$.

⇒: Suppose that $(\forall[\alpha_1,\alpha_2] \in L^I\backslash\{0_{L^I}\})(P_{[\alpha_1,\alpha_2]} = R_{\alpha_1}^{\alpha_2})$, or equivalently that $(\forall[\alpha_1,\alpha_2] \in L^I\backslash\{0_{L^I}\})(x \in P_{[\alpha_1,\alpha_2]} \Leftrightarrow s_x = R(x) \geq_{L^I} [\alpha_1,\alpha_2])$. For all $x \in d_P$, choosing $[\alpha_1,\alpha_2] = s_x = [s_{x,1}, s_{x,2}] \in L^I\backslash\{0_{L^I}\}$ implies $x \in P_{[s_{x,1},s_{x,2}]}$, and thus $s_x \in \{[\alpha_1,\alpha_2] \in L^I\backslash\{0_{L^I}\} \mid x \in P_{[\alpha_1,\alpha_2]}\} = S_x$.

□

The above condition is however not very useful in practice. For a family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I\backslash\{0_{L^I}\}}$ it would be needed to calculate the set S_x for all $x \in d_P$ and to check whether $s_x \in S_x$. To avoid this work, a necessary condition on the sets $P_{[\alpha_1,\alpha_2]}$ can be used.

Proposition 3. *For a decreasing family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I\backslash\{0_{L^I}\}}$ of crisp subsets of $\mathbb{R}^n$, the interval-valued fuzzy set R defined in (6) and the sets d_P and S_x and the supremum s_x of the latter set, respectively defined in expressions (7)-(9), it holds that:*

$$(\forall x \in d_P)(s_x \in S_x) \Rightarrow (\forall[\alpha_1,\alpha_2] \in L^I\backslash\{0_{L^I}\})\Big(P_{[\alpha_1,\alpha_2]} = \bigcap_{[\beta_1,\beta_2]\ll_{L^I}[\alpha_1,\alpha_2]} P_{[\beta_1,\beta_2]}\Big).$$

Proof. Let $[\alpha_1,\alpha_2] \in L^I\backslash\{0_{L^I}\}$. For all $x \in \bigcap_{[\beta_1,\beta_2]\ll_{L^I}[\alpha_1,\alpha_2]} P_{[\beta_1,\beta_2]}$ it holds:

$$\begin{aligned} x \in \bigcap_{[\beta_1,\beta_2]\ll_{L^I}[\alpha_1,\alpha_2]} P_{[\beta_1,\beta_2]} &\Leftrightarrow (\forall[\beta_1,\beta_2] \ll_{L^I} [\alpha_1,\alpha_2])(x \in P_{[\beta_1,\beta_2]}) \\ &\Leftrightarrow (\forall[\beta_1,\beta_2] \ll_{L^I} [\alpha_1,\alpha_2])([\beta_1,\beta_2] \in \\ &\qquad \{[\gamma_1,\gamma_2] \in L^I\backslash\{0_{L^I}\} \mid x \in P_{[\beta_1,\beta_2]}\}) \\ &\Leftrightarrow (\forall[\beta_1,\beta_2] \ll_{L^I} [\alpha_1,\alpha_2]) \\ &\qquad (x \in d_P \text{ and } [\beta_1,\beta_2] \in S_x). \end{aligned}$$

Since it is given that $s_x \in S_x$, it follows from $[\alpha_1, \alpha_2] \leq s_x$ that $[\alpha_1, \alpha_2] \in S_x$, or thus $x \in P_{[\alpha_1,\alpha_2]}$. As a consequence $\bigcap\limits_{[\beta_1,\beta_2] \ll_{L^I} [\alpha_1,\alpha_2]} P_{[\beta_1,\beta_2]} \subseteq P_{[\alpha_1,\alpha_2]}$.

On the other hand, since the family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2] \in L^I \backslash \{0_{L^I}\}}$ is a decreasing family, we have that $(\forall [\beta_1, \beta_2] \in L^I \backslash \{0_{L^I}\})([\beta_1, \beta_2] \ll_{L^I} [\alpha_1, \alpha_2] \Rightarrow P_{[\beta_1,\beta_2]} \supseteq P_{[\alpha_1,\alpha_2]})$ and thus $\bigcap\limits_{[\beta_1,\beta_2] \ll_{L^I} [\alpha_1,\alpha_2]} P_{[\beta_1,\beta_2]} \supseteq P_{[\alpha_1,\alpha_2]}$. $\square$

Example 2. The results in Example 1 could also have been obtained using the above proposition.

Let $[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}$, then it holds that:

$$
\begin{aligned}
& (\forall [\beta_1, \beta_2] \in L^I \backslash \{0_{L^I}\}) \\
& ([\beta_1, \beta_2] \ll_{L^I} [\alpha_1, \alpha_2] \Rightarrow -1 + \alpha_1 < -1 + \beta_1 \text{ and } 1 - \alpha_2 < 1 - \beta_2) \\
\Rightarrow \quad & (\forall [\beta_1, \beta_2] \in L^I \backslash \{0_{L^I}\})([\beta_1, \beta_2] \ll_{L^I} [\alpha_1, \alpha_2] \Rightarrow \\
& (\forall x \in [-1 + \alpha_1, 1 - \alpha_2])(x \in P_{[\beta_1,\beta_2]} =]-1 + \beta_1, 1 - \beta_2[)) \\
\Rightarrow \quad & (\forall x \in [-1 + \alpha_1, 1 - \alpha_2])(x \in \bigcap_{[\beta_1,\beta_2] \ll_{L^I} [\alpha_1,\alpha_2]} P_{[\beta_1,\beta_2]}).
\end{aligned}
$$

So, e.g. $1 - \alpha_2 \in \bigcap\limits_{[\beta_1,\beta_2] \ll_{L^I} [\alpha_1,\alpha_2]} P_{[\beta_1,\beta_2]}$, but $1 - \alpha_2 \notin P_{[\alpha_1,\alpha_2]}$, and thus $P_{[\alpha_1,\alpha_2]} \neq \bigcap\limits_{[\beta_1,\beta_2] \ll_{L^I} [\alpha_1,\alpha_2]} P_{[\beta_1,\beta_2]}$.

$\diamondsuit$

The condition in Proposition 3 is however not a sufficient condition as the following example illustrates.

Example 3. Let $P_{[\alpha_1,\alpha_2]} = [\frac{\alpha_1+\alpha_2}{2}, 1]$ for all $[\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\}$. It holds that $(\forall [\alpha_1, \alpha_2] \in L^I \backslash \{0_{L^I}\})(P_{[\alpha_1,\alpha_2]} = \bigcap\limits_{[\beta_1,\beta_2] \ll_{L^I} [\alpha_1,\alpha_2]} P_{[\beta_1,\beta_2]})$. However, it does not hold that $(\forall x \in d_P)(s_x \in S_x)$. Consider for example the set $S_{0.5}$. $[0.5, 0.5] \in S_{0.5}$ and for all $[\alpha_1, \alpha_2]$, α_1 can not be greater than 0.5 since then

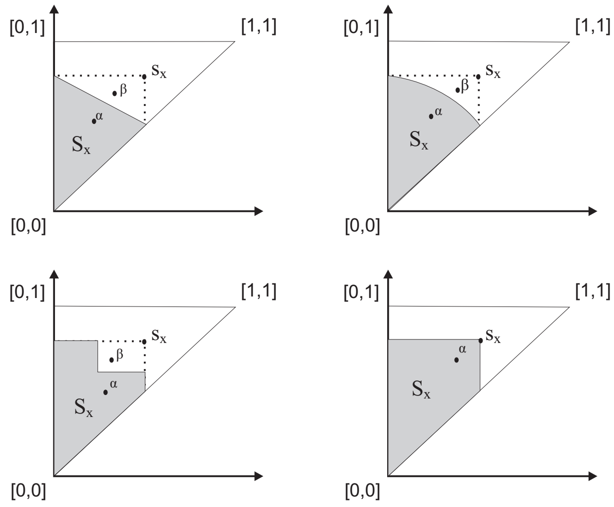

Figure 2: A graphical representation of some possible sets S_x.

$\frac{\alpha_1+\alpha_2}{2} > 0.5$. Further also $[0,1] \in S_{0.5}$, so we can conclude that $\sup S_{0.5} = [0.5,1] \notin S_{0.5}$.

As a consequence it does not hold that $P_{[\alpha_1,\alpha_2]} = R_{\alpha_1}^{\alpha_2}$ for all $[\alpha_1,\alpha_2] \in L^I \backslash \{0_{L^I}\}$. Indeed, $R(0.5) = s_{0.5} = [0.5,1]$ and thus $0.5 \in R_{0.5}^1$ at one hand, but on the other hand $0.5 \notin P_{[0.5,1]} = [0.75,1]$.

$\diamondsuit$

The given condition is not sufficient because is does not necessarily hold that $(\forall[\beta_1,\beta_2] \in L^I \backslash \{0_{L^I}\})([\beta_1,\beta_2] \ll_{L^I} s_x \Rightarrow [\beta_1,\beta_2] \in S_x))$. In the above example, $s_{0.5} = \sup S_{0.5} = [0.5,1]$. So, e.g. $[0.3,0.8] \ll_{L^I} s_{0.5}$, but $[0.3,0.8] \notin S_{0.5}$ since $0.5 \notin P_{[0.3,0.8]} = [0.55,1]$.

Fig. 2 gives a graphical representation of some possible sets S_x. In the first three examples, it does not hold that an interval $\beta \ll s_x$ belongs to the set S_x. In these examples, there can be found an $\alpha \in S_x$, for which it holds that if we keep increasing α_1 or α_2, α will no longer belong to S_x at some point, but still $\alpha \ll s_x$. However, if we then keep decreasing the other bound (α_2 or α_1 respectively) at some point α will again belong to the set S_x. If we want that every interval $\beta \ll s_x$ belongs to S_x, S_x needs to have the form of the fourth example. In that example, for an arbitrary $\alpha \in S_x$, we see that if we keep increasing α_1 or α_2, α will no longer belong to S_x at some point (or reach its maximum possible value). This time however $\alpha \not\ll s_x$ then and decreasing the other bound (α_2 or α_1 respectively) will not result in α belonging to the set S_x again anymore. This special case leads us to the following Lemma.

Lemma 2. *For a decreasing family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\}}$ of crisp subsets of $\mathbb{R}^n$, the interval-valued fuzzy set R defined in (6) and the sets d_P and S_x and the supremum s_x of the latter set, respectively defined in expressions (7)-(9), we have that*

$$(\forall x \in d_P)(\forall t \in L^I\setminus\{0_{L^I}\})(t \ll_{L^I} s_x \Rightarrow t \in S_x)$$
$$\Updownarrow$$
$$[SC] : \Big(\forall[\alpha_1,\alpha_2] \in L^I\setminus\{0_{L^I}\}\Big)\Big(\forall x \in \mathbb{R}^n\Big)\Big(x \notin P_{[\alpha_1,\alpha_2]} \Rightarrow$$
$$\big((\forall[\beta_1,\beta_2] \in L^I\setminus\{0_{L^I})((\beta_1 < \alpha_1 \text{ and } \beta_2 > \alpha_2) \Rightarrow x \notin P_{[\beta_1,\beta_2]})\big) \text{ or}$$
$$\big((\forall[\beta_1,\beta_2] \in L^I\setminus\{0_{L^I})((\beta_1 > \alpha_1 \text{ and } \beta_2 < \alpha_2) \Rightarrow x \notin P_{[\beta_1,\beta_2]})\big)\Big)$$

Proof.

- $\Rightarrow$: Suppose that [SC] does not hold, or thus that it holds that $\Big(\exists[\alpha_1,\alpha_2] \in L^I\setminus\{0_{L^I}\}\Big)\Big(\exists x \in \mathbb{R}^n\Big)\Big(x \notin P_{[\alpha_1,\alpha_2]}$ and $\big((\exists[\beta_1,\beta_2] \in L^I\setminus\{0_{L^I}\})((\beta_1 < \alpha_1$ and $\beta_2 > \alpha_2)$ and $x \in P_{[\beta_1,\beta_2]})\big)$ and $\big((\exists[\gamma_1,\gamma_2] \in L^I\setminus\{0_{L^I}\})((\gamma_1 > \alpha_1$ and $\gamma_2 < \alpha_2)$ and $x \in P_{[\gamma_1,\gamma_2]})\big)\Big)$. This would mean that $s_{x,1} \geq \gamma_1$ and

$s_{x,2} \geq \beta_2$. Further, $[\alpha_1, \alpha_2] \ll_{L^I} [\gamma_1, \beta_2] \leq_{L^I} s_x$ and thus $[\alpha_1, \alpha_2] \in S_x$ and $x \in P_{[\alpha_1,\alpha_2]}$, and hence a contradiction.

$\Leftarrow$: Suppose that the condition [SC] is fulfilled. Let $t \in L^I \backslash \{0_{L^I}\}$, so that $t \ll_{L^I} s_x$. We have to prove that $t \in S_x$.

Suppose that $t \notin S_x$. Since $t \ll_{L^I} s_x$, we have that $t_1 < s_{x,1}$. As a consequence, t_1 is no upperbound for the lower borders of the elements of S_x, which implies that $(\exists y \in]t_1, s_{x,1}])(\exists z \in [y, 1])([y, z] \in S_x)$. If $z \geq t_2$ then we would get a contradiction since then $x \in P_{[y,z]} \subseteq P_{[t_1,t_2]}$ and hence $t \in S_x$. So $z < t_2$ and thus $(\exists [y, z] \in L^I \backslash \{0_{L^I}\})(y > t_1$ and $z < t_2$ and $x \in P_{[y,z]})$.

Analogously, since $t \ll_{L^I} s_x$, we have that $t_2 < s_{x,2}$. As a consequence, t_2 is no upperbound for the lower borders of the elements of S_x, which implies that $(\exists z' \in]t_2, s_{x,2}])(\exists y' \in [0, z'])([y', z'] \in S_x)$. If $y' \geq t_1$ then we would get a contradiction since then $x \in P_{[y',z']} \subseteq P_{[t_1,t_2]}$, i.e., $t \in S_x$. So $y' < t_1$ and thus $(\exists [y', z'] \in L^I \backslash \{0_{L^I}\})(y' < t_1$ and $z' > t_2$ and $x \in P_{[y',z']})$.

If we combine the above results, then we find that it would hold that $x \notin P_{[t_1,t_2]}$ and $(\exists [y, z] \in L^I \backslash \{0_{L^I}\})(y > t_1$ and $z < t_2$ and $x \in P_{[y,z]})$ and $(\exists [y', z'] \in L^I \backslash \{0_{L^I}\})(y' < t_1$ and $z' > t_2$ and $x \in P_{[y',z']})$, and hence a contradiction. So $t \in S_x$.

$\square$

The family defined in Example 1 fulfils the condition [SC]. More general, a decreasing family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2] \in L^I \backslash \{0_{L^I}\}}$ for which $P_{[\alpha_1,\alpha_2]}$ is an interval with lower border $f(\alpha_1)$ and upper border $g(\alpha_2)$, where the functions f and g are respectively increasing and decreasing over $[0,1]$ and $f(\beta_1) \leq g(\beta_2)$ for all $[\beta_1, \beta_2] \in L^I$, is an example of a family that fulfils condition [SC]. An analogous example of a family that fulfils condition [SC] is e.g. the family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\}}$ for which $P_{[\alpha_1,\alpha_2]}$ is an interval with lower border $h(\alpha_2)$ and upper border $i(\alpha_1)$, where the functions h and i are respectively increasing and decreasing over $[0,1]$ and $h(\beta_2) \leq i(\beta_1)$ for all $[\beta_1, \beta_2] \in L^I$. For families for which [SC] holds, Proposition 3 is now also a sufficient condition.

Proposition 4. *For a decreasing family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\}}$ of crisp subsets of $\mathbb{R}^n$ that fulfils the condition [SC], the interval-valued fuzzy set R defined in (6) and the sets d_P and S_x and the supremum s_x of the latter set, respectively defined in expressions (7)-(9), it holds that:*

$$(\forall x \in d_P)(s_x \in S_x) \Leftrightarrow$$
$$(\forall [\alpha_1, \alpha_2] \in L^I\setminus\{0_{L^I}\})(P_{[\alpha_1,\alpha_2]} = \bigcap_{[\beta_1,\beta_2] \ll_{L^I} [\alpha_1,\alpha_2]} P_{[\beta_1,\beta_2]}).$$

Proof.

$\Rightarrow$: Follows from Proposition 3.

$\Leftarrow$: Let $x \in d_P$. Since the family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\}}$ fulfils condition

[SC], Lemma 2 can be used and we obtain successively:

$$
\begin{aligned}
& (\forall[\beta_1,\beta_2] \in L^I\backslash\{0_{L^I}\})([\beta_1,\beta_2] \ll_{L^I} s_x \Rightarrow [\beta_1,\beta_2] \in S_x)) \\
\Leftrightarrow\ & (\forall[\beta_1,\beta_2] \in L^I\backslash\{0_{L^I}\})([\beta_1,\beta_2] \ll_{L^I} s_x \Rightarrow (x \in P_{[\beta_1,\beta_2]})) \\
\Leftrightarrow\ & x \in \bigcap_{[\beta_1,\beta_2]\ll_{L^I} s_x} P_{[\beta_1,\beta_2]} \\
\Leftrightarrow\ & x \in P_{[s_{x,1},s_{x,2}]} \\
\Leftrightarrow\ & s_x \in S_x = \{[\alpha_1,\alpha_2] \in L^I\backslash\{0_{L^I}\} \mid x \in P_{[\alpha_1,\alpha_2]}\}.
\end{aligned}
$$

$\square$

Remark that if a decreasing family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I\backslash\{0_{L^I}\}}$ does not fulfil condition [SC], then it does not hold that $(\forall[\alpha_1,\alpha_2] \in L^I\backslash\{0_{L^I}\})(P_{[\alpha_1,\alpha_2]} = R_{\alpha_1}^{\alpha_2})$ anyway. Indeed, due to Lemma 2 it does not hold that $(\forall x \in d_P)(\forall t \in L^I\backslash\{0_{L^I}\})(t \ll_{L^I} s_x \Rightarrow t \in S_x)$ and thus $(\exists x \in d_P)(\exists t \in L^I\backslash\{0_{L^I}\})(t \ll_{L^I} s_x$ and $t \notin S_x)$. This implies that $x \notin P_{[t_1,t_2]} \supseteq P_{[s_{x,1},s_{x,2}]}$ or thus $s_x \notin S_x$.

Example 4. The family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I\backslash\{0_{L^I}\}}$ of crisp subsets of $\mathbb{R}^n$, given by $P_{[\alpha_1,\alpha_2]} = [-1+\alpha_1, 1-\alpha_2]$ for all $[\alpha_1,\alpha_2] \in L^I\backslash\{0_{L^I}\}$, is an example of a family for which $(\forall[\alpha_1,\alpha_2] \in L^I\backslash\{0_{L^I}\})(P_{[\alpha_1,\alpha_2]} = R_{\alpha_1}^{\alpha_2})$, with the interval-valued fuzzy set R as defined in (6).

$\diamondsuit$

3.2.2. The Construction Principle

Based on the results from subsection 3.2.1, we can extend an increasing operator ϕ on $\mathcal{P}(\mathbb{R}^n)$ (i.e. the set of all crisp subsets of $\mathbb{R}^n$) to an operator Φ on $\mathcal{IVFS}(\mathbb{R}^n)$ as follows:

$$\Phi(A) = \bigcup_{[\alpha_1,\alpha_2]\in L^I\backslash\{0_{L^I}\}} [\alpha_1,\alpha_2]\phi(A_{\alpha_1}^{\alpha_2}), \text{ for all } A \in \mathcal{IVFS}(\mathbb{R}^n).$$

Operators having two or more arguments can be extended analogously. We illustrate this for an increasing operator ψ on $\mathcal{P}(\mathbb{R}^n)\times\mathcal{P}(\mathbb{R}^n)$ (like the binary dilation):

$$\Psi(A,B) = \bigcup_{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\}} [\alpha_1,\alpha_2]\psi(A_{\alpha_1}^{\alpha_2}, B_{\alpha_1}^{\alpha_2}), \text{ for all } A,B\in\mathcal{IVFS}(\mathbb{R}^n).$$

As discussed in the introduction of this subsection, for the operators Φ and Ψ it does not necessarily hold that:

$$(\forall[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\})(\Phi(A)_{\alpha_1}^{\alpha_2} = \phi(A_{\alpha_1}^{\alpha_2}))$$
$$(\forall[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\})(\Psi(A,B)_{\alpha_1}^{\alpha_2} = \psi(A_{\alpha_1}^{\alpha_2}, B_{\alpha_1}^{\alpha_2})).$$

Applied to the binary morphological operators, we have the following definitions:

Definition 8. *A binary morphological operator M is called increasing (w.r.t. the image) if for binary sets $A_1, A_2, B \in \mathbb{R}^n$, with $A_1 \subseteq A_2$, then $M(A_1,B)\subseteq M(A_2,B)$.*

An binary morphological operator M is called increasing w.r.t. the structuring element if for binary sets $A, B_1, B_2 \in \mathbb{R}^n$, with $B_1\subseteq B_2$, then $M(A,B_1)\subseteq M(A,B_2)$.

It can be shown that the binary dilation, erosion, opening and closing are all increasing (w.r.t. the image). However, only the dilation is increasing w.r.t. to both the image and structuring element and can be extended to interval-valued fuzzy sets by the help of the above introduced construction principle.

Let $A, B \in \mathcal{IVFS}(\mathbf{R}^n)$. Using the construction principle, we define the extended dilation $\widetilde{D(A,B)}$ of A by B as follows:

$$\widetilde{D(A,B)} = \bigcup_{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\}} [\alpha_1,\alpha_2] D(A_{\alpha_1}^{\alpha_2}, B_{\alpha_1}^{\alpha_2}). \tag{10}$$

It turns out that the constructed interval-valued fuzzy dilation corresponds to the interval-valued fuzzy dilation based on the extended minimum operator $\mathcal{C}_{\min}$.

Proposition 5. *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then for all $y \in \mathbb{R}^n$ it holds that:*

$$\widetilde{D(A,B)}(y) = \sup_{x\in T_y(-d_B)\cap d_A} \mathcal{C}_{\min}(B(y-x), A(x)) = D^I_{\mathcal{C}_{\min}}(A,B)(y).$$

Proof. Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, and let $y \in \mathbb{R}^n$. From the definition of the binary dilation,

$$D(A_{\alpha_1}^{\alpha_2}, B_{\alpha_1}^{\alpha_2})(y) = \begin{cases} 1 & \text{if } y \in D(A_{\alpha_1}^{\alpha_2}, B_{\alpha_1}^{\alpha_2}) \\ 0 & \text{else} \end{cases},$$

it follows that:

$$\begin{aligned}
\widetilde{D(A,B)}(y) &= \sup_{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\}} ([\alpha_1,\alpha_2]D(A^{\alpha_2}_{\alpha_1}, B^{\alpha_2}_{\alpha_1}))(y) \\
&= \sup\{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\} \mid y \in D(A^{\alpha_2}_{\alpha_1}, B^{\alpha_2}_{\alpha_1})\} \\
&= \sup\{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\} \mid T_y(-B^{\alpha_2}_{\alpha_1}) \cap A^{\alpha_2}_{\alpha_1} \neq \emptyset\} \\
&= \sup\{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\} \mid (\exists x \in T_y(-d_B)\cap d_A) \\
&\quad (x \in T_y(-B^{\alpha_2}_{\alpha_1}) \text{ and } x \in A^{\alpha_2}_{\alpha_1})\} \\
&= \sup\{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\} \mid (\exists x \in T_y(-d_B)\cap d_A) \\
&\quad ((B_1(y-x)\geq \alpha_1 \text{ and } A_1(x)\geq \alpha_1) \text{ and} \\
&\quad (B_2(y-x)\geq \alpha_2 \text{ and } A_2(x)\geq \alpha_2))\} \\
&= \sup\{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\} \mid (\exists x \in T_y(-d_B)\cap d_A) \\
&\quad (\mathcal{C}_{\min}(B(y-x),A(x)) \geq_{L^I} [\alpha_1,\alpha_2])\} \\
&\equiv (*).
\end{aligned}$$

We have to prove that $(*)$ is equal to

$$\sup\{\mathcal{C}_{\min}(B(y-x),A(x)) \mid x \in T_y(-d_B)\cap d_A\} \equiv (**).$$

- It holds that:

$$\begin{aligned}
(*) &= \sup\{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\} \mid (\exists x \in T_y(-d_B)\cap d_A) \\
&\qquad ([\alpha_1,\alpha_2] \leq_{L^I} \mathcal{C}_{\min}(B(y-x),A(x)))\} \\
&\leq_{L^I} \sup\{[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\} \mid \\
&\qquad ([\alpha_1,\alpha_2] \leq_{L^I} \sup_{x\in T_y(-d_B)\cap d_A} \mathcal{C}_{\min}(B(y-x),A(x)))\} \\
&= \sup_{x\in T_y(-d_B)\cap d_A} \mathcal{C}_{\min}(B(y-x),A(x))) \\
&= (**)
\end{aligned}$$

- On the other hand also $(**) \leq_{L^I} (*)$. If $T_y(-d_B) \cap d_A = \emptyset$, then $(**) = 0_{L^I}$ and thus $(**) \leq_{L^I} (*)$. Otherwise, consider an arbitrary $\epsilon \gg_{L^I} 0_{L^I}$. Then it holds that:

$$
\begin{aligned}
& (**)_1 - \epsilon_1 \text{ and } (**)_2 - \epsilon_2 \text{ are no upper bound for respectively} \\
& \text{the lower and the upper bounds of the intervals in the set} \\
& \{\mathcal{C}_{\min}(B(y-x), A(x)) \mid x \in T_y(-d_B) \cap d_A\}. \\
\Rightarrow\ & (\exists x \in T_y(-d_B) \cap d_A)((**)_1 - \epsilon_1 < \mathcal{C}_{\min}(B(y-x), A(x))_1) \text{ and} \\
& (\exists x' \in T_y(-d_B) \cap d_A)((**)_2 - \epsilon_2 < \mathcal{C}_{\min}(B(y-x'), A(x'))_2) \\
\Rightarrow\ & (**)_1 - \epsilon_1 \in \{\alpha_1 \mid (\exists \alpha_2 \in [\alpha_1, 1] \text{ such that } [\alpha_1, \alpha_2] \in L^I \setminus \{0_{L^I}\}) \\
& \text{and } (\exists x \in T_y(-d_B) \cap d_A)([\alpha_1, \alpha_2] \leq_{L^I} \mathcal{C}_{\min}(B(y-x), A(x)))\} \text{ and} \\
& (**)_2 - \epsilon_2 \in \{\alpha_2 \mid (\exists \alpha_1 \in [0, \alpha_2] \text{ such that } [\alpha_1, \alpha_2] \in L^I \setminus \{0_{L^I}\}) \\
& \text{and } (\exists x \in T_y(-d_B) \cap d_A)([\alpha_1, \alpha_2] \leq_{L^I} \mathcal{C}_{\min}(B(y-x), A(x)))\} \\
\Rightarrow\ & (**)_1 - \epsilon_1 \leq \sup\{\alpha_1 \mid (\exists \alpha_2 \in [\alpha_1, 1] \text{ such that } [\alpha_1, \alpha_2] \in L^I \setminus \{0_{L^I}\}) \\
& \text{and } (\exists x \in T_y(-d_B) \cap d_A)([\alpha_1, \alpha_2] \leq_{L^I} \mathcal{C}_{\min}(B(y-x), A(x)))\} \text{ and} \\
& (**)_2 - \epsilon_2 \leq \sup\{\alpha_2 \mid (\exists \alpha_1 \in [0, \alpha_2] \text{ such that } [\alpha_1, \alpha_2] \in L^I \setminus \{0_{L^I}\}) \\
& \text{and } (\exists x \in T_y(-d_B) \cap d_A)([\alpha_1, \alpha_2] \leq_{L^I} \mathcal{C}_{\min}(B(y-x), A(x)))\} \\
\Rightarrow\ & [(**)_1 - \epsilon_1, (**)_2 - \epsilon_2] \leq_{L^I} \sup\{[\alpha_1, \alpha_2] \in L^I \setminus \{0_{L^I}\} \mid \\
& (\exists x \in T_y(-d_B) \cap d_A)([\alpha_1, \alpha_2] \leq_{L^I} \mathcal{C}_{\min}(B(y-x), A(x)))\}
\end{aligned}
$$

  Taking $\epsilon \to 0_{L^I}$ gives the result.

$\square$

Since the binary erosion is not increasing w.r.t. the structuring element, we cannot use the construction principle to extend this morphological operator to an interval-valued fuzzy morphological operator. Such interval-valued fuzzy erosion can however be constructed by duality properties [24] (where $(co_{\mathcal{N}}A)(x) = \mathcal{N}(A(x))$, for all $A \in \mathcal{IVFS}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$):

$$E^I_{\mathcal{I}_{\min},\mathcal{N}_s}(A,B)(y) \quad = \quad (co_{\mathcal{N}_s}(D^I_{\mathcal{C}_{\min}}(co_{\mathcal{N}_s}(A), -B)))(y)$$

The interval-valued fuzzy opening and closing can then be constructed as a combination of the interval-valued fuzzy dilation and erosion.

3.3. Construction based on strict $[\alpha_1, \alpha_2]$-cuts

In the previous subsection we developed a framework to construct an interval-valued fuzzy dilation using binary dilations of weak $[\alpha_1, \alpha_2]$-cuts. In this subsection we investigate whether this can also be achieved by using strict $[\alpha_1, \alpha_2]$-cuts. Because of the similarity to the case of weak $[\alpha_1, \alpha_2]$-cuts, we will leave the proofs in this subsection to the reader.

3.3.1. Introduction

By using the interval-valued fuzzy sets $[\alpha_1, \alpha_2]A^{\alpha_2}_{\alpha_1}$, based on the strict cuts of an interval-valued fuzzy set $A \in \mathcal{IVFS}(\mathbb{R}^n)$, the original interval-valued fuzzy set A can be reconstructed as follows.

Lemma 3. *Let $A \in \mathcal{IVFS}(\mathbb{R}^n)$. It holds that $A = \bigcup_{[\alpha_1,\alpha_2]\in L^I\setminus U_{L^I}} [\alpha_1, \alpha_2]A^{\overline{\alpha_2}}_{\overline{\alpha_1}}$, i.e.:*

$$A(x) = \sup_{[\alpha_1,\alpha_2]\in L^I\setminus U_{L^I}} ([\alpha_1, \alpha_2]A^{\overline{\alpha_2}}_{\overline{\alpha_1}})(x) = \sup\{[\alpha_1, \alpha_2] \in L^I\setminus U_{L^I} \mid x \in A^{\overline{\alpha_2}}_{\overline{\alpha_1}}\},\ \forall x \in \mathbb{R}^n.$$

If we now consider a family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2]\in L^I\backslash U_{L^I}}$ of crisp subsets of $\mathbb{R}^n$ that is decreasing ($[\alpha_1,\alpha_2] \leq_{L^I} [\alpha_3,\alpha_4] \Rightarrow Q_{\overline{[\alpha_1,\alpha_2]}} \supseteq Q_{\overline{[\alpha_3,\alpha_4]}}$) and we define the interval-valued fuzzy set V in $\mathbb{R}^n$ for all $x \in \mathbb{R}^n$ as,

$$\begin{aligned} V(x) &= \sup_{[\alpha_1,\alpha_2]\in L^I\backslash U_{L^I}} ([\alpha_1,\alpha_2]Q_{\overline{[\alpha_1,\alpha_2]}})(x) \\ &= \sup\{[\alpha_1,\alpha_2] \in L^I\backslash U_{L^I} \mid x \in Q_{\overline{[\alpha_1,\alpha_2]}}\}, \end{aligned} \tag{11}$$

then we might wonder whether it holds that $(\forall[\alpha_1,\alpha_2] \in L^I\backslash U_{L^I})(V_{\overline{\alpha_1}}^{\overline{\alpha_2}} = Q_{\overline{[\alpha_1,\alpha_2]}})$. In contrast to the case of weak $[\alpha_1,\alpha_2]$-cuts, there is no inclusion that always holds.

Example 5. Let $Q_{\overline{[\alpha_1,\alpha_2]}} = [\frac{\alpha_1+\alpha_2}{2}, 1]$ for all $[\alpha_1,\alpha_2] \in L^I\backslash U_{L^I}$. Consider e.g. $x = 0.4$. $x \in Q_{\overline{[0.4,0.4]}}$ and α_1 can not be greater than 0.4 since then $\frac{\alpha_1+\alpha_2}{2} > 0.4$. Further, $0.4 \in Q_{\overline{[0,\alpha_2]}}$, for all $\alpha_2 \leq 0.8$. So, $V(0.4) = [0.4, 0.8]$ and thus $0.4 \in V_{\overline{0.3}}^{\overline{0.7}}$ at one hand, but on the other hand $0.4 \notin Q_{\overline{[0.3,0.7]}} = [0.5, 1]$. As a consequence, it does not hold for all $[\alpha_1,\alpha_2] \in L^I\backslash U_{L^I}$ that $Q_{\overline{[\alpha_1,\alpha_2]}} \supseteq V_{\overline{\alpha_1}}^{\overline{\alpha_2}}$.

Neither does it hold for all $[\alpha_1,\alpha_2] \in L^I\backslash U_{L^I}$ that $Q_{\overline{[\alpha_1,\alpha_2]}} \subseteq V_{\overline{\alpha_1}}^{\overline{\alpha_2}}$. For every $[\alpha_1,\alpha_2] \in L^I\backslash U_{L^I}$, we have for $[\beta_1,\beta_2] \ll_{L^I} [\alpha_1,\alpha_2]$ that $\frac{\beta_1+\beta_2}{2} < \frac{\alpha_1+\alpha_2}{2}$ or thus $\frac{\alpha_1+\alpha_2}{2} \in Q_{\overline{[\beta_1,\beta_2]}}$. As a consequence $V(\frac{\alpha_1+\alpha_2}{2}) = \sup\{[\beta_1,\beta_2] \in L^I\backslash U_{L^I} \mid \frac{\alpha_1+\alpha_2}{2} \in Q_{\overline{[\beta_1,\beta_2]}}\} = [\alpha_1,\alpha_2]$ or thus $\frac{\alpha_1+\alpha_2}{2} \notin V_{\overline{\alpha_1}}^{\overline{\alpha_2}}$. On the other hand $\frac{\alpha_1+\alpha_2}{2} \in Q_{\overline{[\alpha_1,\alpha_2]}} = [\frac{\alpha_1+\alpha_2}{2}, 1]$ what means that $V_{\overline{\alpha_1}}^{\overline{\alpha_2}} \not\supseteq Q_{\overline{[\alpha_1,\alpha_2]}}$.

$\diamondsuit$

The equality holds however under certain conditions. To formulate these conditions, we define the set d_Q as

$$d_Q = \{x \in \mathbb{R}^n \mid (\exists[\alpha_1,\alpha_2] \in L^I\backslash U_{L^I})(x \in Q_{\overline{[\alpha_1,\alpha_2]}})\}. \tag{12}$$

Further, for a fixed point $x \in d_Q$, we introduce the set T_x, given by

$$T_x = \{[\alpha_1, \alpha_2] \in L^I \backslash U_{L^I} \mid x \in Q_{\overline{[\alpha_1,\alpha_2]}}\}, \tag{13}$$

and we denote the supremum of this set by $t_x = [t_{x,1}, t_{x,2}]$:

$$t_x = \sup T_x. \tag{14}$$

Remark that $T_x \neq \emptyset$.

The following Proposition gives a necessary condition such that the equality holds:

Proposition 6. *For a decreasing family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2] \in L^I \backslash U_{L^I}}$ of crisp subsets of $\mathbb{R}^n$, the interval-valued fuzzy set V defined in (11) and the sets d_Q and T_x and the supremum t_x of the latter set, respectively defined in expressions (12)-(14), it holds that:*

$$(\forall [\alpha_1, \alpha_2] \in L^I \backslash U_{L^I})(Q_{\overline{[\alpha_1,\alpha_2]}} = V_{\overline{\alpha_1}}^{\overline{\alpha_2}}) \Rightarrow$$
$$(\forall x \in d_Q)(\forall [\beta_1, \beta_2] \in T_x)(t_x \gg_{L^I} [\beta_1, \beta_2]).$$

We would like to mention here that the condition $(\forall [\beta_1, \beta_2] \in T_x)(t_x \gg_{L^I} [\beta_1, \beta_2]) \Rightarrow t_x \notin T_x$ is a neccessary and sufficient condition such that it would hold that $(\forall [\alpha_1, \alpha_2] \in L^I \backslash U_{L^I}\})(Q_{\overline{[\alpha_1,\alpha_2]}} \subseteq V_{\overline{\alpha_1}}^{\overline{\alpha_2}})$.

Proposition 7. *For a decreasing family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2] \in L^I \backslash U_{L^I}}$ of crisp subsets of $\mathbb{R}^n$, the interval-valued fuzzy set V defined in (11) and the sets d_Q and T_x and the supremum t_x of the latter set, respectively defined in expressions (12)-(14), it holds that:*

$$(\forall [\alpha_1, \alpha_2] \in L^I \backslash U_{L^I})(Q_{\overline{[\alpha_1,\alpha_2]}} \subseteq V_{\overline{\alpha_1}}^{\overline{\alpha_2}}) \Leftrightarrow$$
$$(\forall x \in d_Q)(\forall [\beta_1, \beta_2] \in T_x)(t_x \gg_{L^I} [\beta_1, \beta_2]).$$

The condition in Proposition 6 is however not a sufficient condition for the equality to hold as the following example illustrates.

Example 6. Let $Q_{\overline{[\alpha_1,\alpha_2]}} =]\frac{\alpha_1+\alpha_2}{2}, 1]$ for all $[\alpha_1, \alpha_2] \in L^I \backslash U_{L^I}$. Then $d_Q =]0, 1]$. For $x \in d_Q$, it holds that $[\beta_1, \beta_2] \in T_x \Leftrightarrow x \in]\frac{\beta_1+\beta_2}{2}, 1]$, which is equivalent to $\frac{\beta_1+\beta_2}{2} < x$. So $[\beta_1, \beta_1] \in T_x$ for all $\beta_1 < x$. It is impossible that $\beta_1 \geq x$ for any $[\beta_1, \beta_2] \in T_x$, since then $\frac{\beta_1+\beta_2}{2} \geq x$. So the first component of each element of T_x is smaller than the first component of the supremum of T_x $(= x)$. Further, also $[0, y] \in T_x$ for all y such that $y < 2x$ and $y < 1$. It is impossible that $\beta_2 \geq 2x$ or $\beta_2 \geq 1$ for any $[\beta_1, \beta_2] \in T_x$, since then respectively $\frac{\beta_1+\beta_2}{2} \geq x$ and $[\beta_1, \beta_2] \notin L^I \backslash U_{L^I}$. So the second component of each element of T_x is smaller than the second component of the supremum of T_x $(= \min(2x, 1))$. We conclude that $(\forall x \in d_Q)(\forall [\beta_1, \beta_2] \in T_x)(t_x \gg_{L^I} [\beta_1, \beta_2])$.

It does however not hold that $(\forall [\alpha_1, \alpha_2] \in L^I \backslash U_{L^I})(Q_{\overline{[\alpha_1,\alpha_2]}} = V_{\overline{\alpha_1}}^{\overline{\alpha_2}})$. Consider e.g. $x = 0.4$. $x \in Q_{\overline{[\alpha_1,\alpha_1]}}$ for all $\alpha_1 < 0.4$ and α_1 can not be greater or equal to 0.4 since then $\frac{\alpha_1+\alpha_2}{2} \geq 0.4$. Further, $0.4 \in Q_{\overline{[0,\alpha_2]}}$, for all $\alpha_2 < 0.8$. So, $V(0.4) = [0.4, 0.8]$ and thus $0.4 \in V_{\overline{0.3}}^{\overline{0.7}}$ at one hand, but on the other hand $0.4 \notin Q_{\overline{[0.3,0.7]}} =]0.5, 1]$. As a consequence, it does not hold for all $[\alpha_1, \alpha_2] \in L^I \backslash U_{L^I}$ that $Q_{\overline{[\alpha_1,\alpha_2]}} = V_{\overline{\alpha_1}}^{\overline{\alpha_2}}$.

$\diamondsuit$

The given condition is not sufficient because it does not necessarily hold that $(\forall [\beta_1, \beta_2] \in L^I \backslash U_{L^I})([\beta_1, \beta_2] \ll_{L^I} t_x \Rightarrow [\beta_1, \beta_2] \in T_x))$. In the above example, $t_{0.4} = \sup T_{0.4} = [0.4, 0.8]$. So, e.g. $[0.3, 0.7] \ll_{L^I} t_{0.4}$, but $[0.3, 0.7] \notin T_{0.4}$ since $0.4 \notin Q_{\overline{[0.3,0.7]}} =]0.5, 1]$.

Analogously to Lemma 2 the property $(\forall [\beta_1, \beta_2] \in L^I \backslash U_{L^I})([\beta_1, \beta_2] \ll_{L^I}$

$t_x \Rightarrow [\beta_1, \beta_2] \in T_x)$) does however hold in the following special case:

Lemma 4. *For a decreasing family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2]\in L^I\setminus U_{L^I}}$ of crisp subsets of $\mathbb{R}^n$, the interval-valued fuzzy set V defined in (11) and the sets d_Q and T_x and the supremum t_x of the latter set, respectively defined in expressions (12)-(14), we have that*

$$(\forall x \in d_Q)(\forall r \in L^I\setminus U_{L^I})(r \ll_{L^I} t_x \Rightarrow r \in T_x)$$

$$\Updownarrow$$

$$[SC'] : \Big(\forall[\alpha_1, \alpha_2] \in L^I\setminus U_{L^I}\Big)\Big(\forall x \in \mathbb{R}^n\Big)\Big(x \notin Q_{\overline{[\alpha_1,\alpha_2]}} \Rightarrow$$
$$\big((\forall[\beta_1, \beta_2] \in L^I\setminus U_{L^I})((\beta_1 < \alpha_1 \textit{ and } \beta_2 > \alpha_2) \Rightarrow x \notin Q_{\overline{[\beta_1,\beta_2]}})\big) \textit{ or}$$
$$\big((\forall[\beta_1, \beta_2] \in L^I\setminus U_{L^I})((\beta_1 > \alpha_1 \textit{ and } \beta_2 < \alpha_2) \Rightarrow x \notin Q_{\overline{[\beta_1,\beta_2]}})\big)\Big)$$

Remark that if a decreasing family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2]\in L^I\setminus U_{L^I}}$ of crisp subsets of $\mathbb{R}^n$ does not satisfy condition [SC'], then it will also not hold that $(\forall[\alpha_1, \alpha_2] \in L^I\setminus U_{L^I})(Q_{\overline{[\alpha_1,\alpha_2]}} = V_{\overline{\alpha_1}}^{\overline{\alpha_2}})$. Indeed, if [SC'] does not hold, then $(\exists[\alpha_1, \alpha_2] \in L^I\setminus U_{L^I})(\exists x \in \mathbb{R}^n)(x \notin Q_{\overline{[\alpha_1,\alpha_2]}}$ and $(\exists[\beta_1, \beta_2] \in L^I\setminus U_{L^I})(\beta_1 < \alpha_1$ and $\beta_2 > \alpha_2$ and $x \in Q_{\overline{[\beta_1,\beta_2]}})$ and $(\exists[\gamma_1, \gamma_2] \in L^I\setminus U_{L^I})(\gamma_1 > \alpha_1$ and $\gamma_2 < \alpha_2$ and $x \in Q_{\overline{[\gamma_1,\gamma_2]}}))$. This would mean that $V_1(x) = t_{x,1} \geq \gamma_1 > \alpha_1$ and $V_2(x) = t_{x,2} \geq \beta_2 > \alpha_2$. As a consequence, $x \in V_{\overline{\alpha_1}}^{\overline{\alpha_2}}$ and $x \notin Q_{\overline{[\alpha_1,\alpha_2]}}$.

In what follows we will therefore concentrate on families for which [SC'] holds.

For a decreasing family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2]\in L^I\setminus U_{L^I}}$ of crisp subsets of $\mathbb{R}^n$ for which condition [SC'] does hold, the necessary condition in Proposition 6 becomes a sufficient condition.

Proposition 8. *For a decreasing family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2]\in L^I\setminus U_{L^I}}$ of crisp subsets of $\mathbb{R}^n$ that fulfils condition [SC'], the interval-valued fuzzy set V defined* *in (11) and the sets d_Q and T_x and the supremum t_x of the latter set, respectively defined in expressions (12)-(14), it holds that:*

$$(\forall[\alpha_1,\alpha_2] \in L^I \backslash U_{L^I})(Q_{\overline{[\alpha_1,\alpha_2]}} = V_{\overline{\alpha_1}}^{\overline{\alpha_2}}) \Leftrightarrow$$
$$(\forall x \in d_Q)(\forall[\beta_1,\beta_2] \in T_x)(t_x \gg_{L^I} [\beta_1,\beta_2]).$$

The condition in Proposition 8 is however not always efficient in practice. For a family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2]\in L^I\backslash U_{L^I}}$ that satisfies condition [SC'], it would be needed to calculate the set T_x for all $x \in d_Q$ and to check whether $t_x \gg_{L^I} [\beta_1,\beta_2]$ for all $[\beta_1,\beta_2] \in T_x$. To facilitate this work, an equivalent condition on the sets $Q_{\overline{[\alpha_1,\alpha_2]}}$ can be used.

Proposition 9. *For a decreasing family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2]\in L^I\backslash U_{L^I}}$ of crisp subsets of $\mathbb{R}^n$ that satisfies condition [SC'], the sets d_Q and T_x and the supremum t_x of the latter set, respectively defined in expressions (12)-(14), it holds that:*

$$(\forall x \in d_Q)(\forall[\beta_1,\beta_2] \in T_x)(t_x \gg_{L^I} [\beta_1,\beta_2]) \Leftrightarrow$$
$$(\forall[\alpha_1,\alpha_2] \in L^I \backslash U_{L^I})(Q_{\overline{[\alpha_1,\alpha_2]}} = \bigcup_{[\beta_1,\beta_2]\gg_{L^I}[\alpha_1,\alpha_2]} Q_{\overline{[\beta_1,\beta_2]}}).$$

Example 7. The family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2]\in L^I\backslash U_{L^I}}$ of crisp subsets of $\mathbb{R}^n$, given by $Q_{\overline{[\alpha_1,\alpha_2]}} = [-1+\alpha_1, 1-\alpha_2]$ for all $[\alpha_1,\alpha_2] \in L^I\backslash U_{L^I}$, is an example of a family that satisfies condition [SC'], but for which it does not hold that $(\forall[\alpha_1,\alpha_2] \in L^I\backslash U_{L^I})(Q_{\overline{[\alpha_1,\alpha_2]}} = V_{\overline{\alpha_1}}^{\overline{\alpha_2}})$, with the interval-valued fuzzy set V as defined in (11). Indeed, let $[\alpha_1,\alpha_2] \in L^I\backslash U_{L^I}$, then it holds that $(\forall[\beta_1,\beta_2] \in L^I\backslash U_{L^I})([\beta_1,\beta_2] \gg_{L^I} [\alpha_1,\alpha_2] \Rightarrow -1+\beta_1 > -1+\alpha_1$ and $1-\alpha_2 > 1-\beta_2)$ or thus $(\forall[\beta_1,\beta_2] \in L^I\backslash U_{L^I})([\beta_1,\beta_2] \gg_{L^I} [\alpha_1,\alpha_2] \Rightarrow -1+\alpha_1 \notin Q_{\overline{[\beta_1,\beta_2]}}$ and $1-\alpha_2 \notin Q_{\overline{[\beta_1,\beta_2]}})$. On the other hand $-1+\alpha_1 \in Q_{\overline{[\alpha_1,\alpha_2]}}$ and

$1-\alpha_2 \in Q_{\overline{[\alpha_1,\alpha_2]}}$. So $\bigcup\limits_{[\beta_1,\beta_2] \gg_{L^I} [\alpha_1,\alpha_2]} Q_{\overline{[\beta_1,\beta_2]}} \neq Q_{\overline{[\alpha_1,\alpha_2]}}$.

$\diamondsuit$

Example 8. The family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2] \in L^I \backslash U_{L^I}}$ of crisp subsets of $\mathbb{R}^n$, given by $Q_{\overline{[\alpha_1,\alpha_2]}} =]-1+\alpha_1, 1-\alpha_2[$ for all $[\alpha_1,\alpha_2] \in L^I \backslash U_{L^I}$, is an example of a family for which $(\forall [\alpha_1,\alpha_2] \in L^I \backslash U_{L^I})(Q_{\overline{[\alpha_1,\alpha_2]}} = V_{\overline{\alpha_1}}^{\overline{\alpha_2}})$, with the interval-valued fuzzy set V as defined in (11).

$\diamondsuit$

3.3.2. The Construction Principle

Based on the results from subsection 3.3.1 and analogous to the construction principle based on weak $[\alpha_1,\alpha_2]$-cuts, an increasing operator ϕ on $\mathcal{P}(\mathbb{R}^n)$ can be extended to an operator Φ on $\mathcal{IVFS}(\mathbb{R}^n)$ as follows:

$$\Phi(A) = \bigcup_{[\alpha_1,\alpha_2] \in L^I \backslash U_{L^I}} [\alpha_1,\alpha_2]\phi(A_{\overline{\alpha_1}}^{\overline{\alpha_2}}), \text{ for all } A \in \mathcal{IVFS}(\mathbb{R}^n).$$

Operators having two or more arguments can be extended analogously. We illustrate this for an increasing operator ψ on $\mathcal{P}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n)$ (like the binary dilation):

$$\Psi(A,B) = \bigcup_{[\alpha_1,\alpha_2] \in L^I \backslash U_{L^I}} [\alpha_1,\alpha_2]\psi(A_{\overline{\alpha_1}}^{\overline{\alpha_2}}, B_{\overline{\alpha_1}}^{\overline{\alpha_2}}), \text{ for all } A, B \in \mathcal{IVFS}(\mathbb{R}^n).$$

As discussed in the introduction of this subsection, for the operators Φ and Ψ it does not necessarily hold that:

$$(\forall [\alpha_1,\alpha_2] \in L^I \backslash U_{L^I})(\Phi(A)_{\overline{\alpha_1}}^{\overline{\alpha_2}} = \phi(A_{\overline{\alpha_1}}^{\overline{\alpha_2}}))$$
$$(\forall [\alpha_1,\alpha_2] \in L^I \backslash U_{L^I})(\Psi(A,B)_{\overline{\alpha_1}}^{\overline{\alpha_2}} = \psi(A_{\overline{\alpha_1}}^{\overline{\alpha_2}}, B_{\overline{\alpha_1}}^{\overline{\alpha_2}})).$$

We now extend the binary dilation to interval-valued fuzzy sets by the help of the above introduced construction principle as follows:

Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$. The extended dilation $\widetilde{D(A,B)}'$ of A by B is then given by:

$$\widetilde{D(A,B)}' = \bigcup_{[\alpha_1,\alpha_2] \in L^I \setminus U_{L^I}} [\alpha_1, \alpha_2] D(A_{\overline{\alpha_1}}^{\overline{\alpha_2}}, B_{\overline{\alpha_1}}^{\overline{\alpha_2}}). \tag{15}$$

In constrast to the constructed interval-valued fuzzy dilation based on weak $[\alpha_1, \alpha_2]$-cuts, the constructed dilation based on strict $[\alpha_1, \alpha_2]$-cuts does not necessarily correspond to the interval-valued fuzzy dilation based on the conjunctor $\mathcal{C}_{\min}$. This will however be the case under a specific condition that is not that hard to satisfy in practice.

Proposition 10. *Let $A, B \in \mathcal{IVFS}(\mathbb{R}^n)$, then for all $y \in \mathbb{R}^n$ it holds that:*

$$\widetilde{D(A,B)}'(y) \leq_{L^I} \sup_{x \in T_y(-d_B) \cap d_A} \mathcal{C}_{\min}(B(y-x), A(x)) = D^I_{\mathcal{C}_{\min}}(A,B)(y).$$

If $A(x) \gg_{L^I} 0_{L^I}$, $\forall x \in d_A$ and $B(x) \gg_{L^I} 0_{L^I}$, $\forall x \in d_B$, then

$$\widetilde{D(A,B)}'(y) = \sup_{x \in T_y(-d_B) \cap d_A} \mathcal{C}_{\min}(B(y-x), A(x)) = D^I_{\mathcal{C}_{\min}}(A,B)(y).$$

If either one of the properties $A(x) \gg_{L^I} 0_{L^I}$, $\forall x \in d_A$ or $B(x) \gg_{L^I} 0_{L^I}$, $\forall x \in d_B$ is not satisfied, then $(*)$ is not necessarily equal to $(**)$, as the following example illustrates.

Example 9. Let $A(0) = [0, 0.7]$, $A(x) = [0.3, 0.5]$, $\forall x \in]0,1]$ and $B(x) = [0.2, 0.6], \forall x \in [-0.5, 0]$. Let $y = 0$, then $T_0(-d_B) \cap d_A = [0, 0.5]$. $\forall x \in]0, 0.5]$, $\mathcal{C}_{\min}(B(0-x), A(x)) = [0.2, 0.5]$. For $x = 0$, $\mathcal{C}_{\min}(B(0), A(0)) = [0, 0.6]$. As a consequence $\sup\{\mathcal{C}_{\min}(B(y-x), A(x)) \mid x \in T_y(-d_B) \cap d_A\} = \sup\{[0, 0.6], [0.2, 0.5]\} = [0.2, 0.6]$. On the other hand, there does not exist an $[\alpha_1, \alpha_2] \in L^I$ such that $[\alpha_1, \alpha_2] \ll_{L^I} [0, 0.6]$ and $(\forall [\alpha_1, \alpha_2] \in L^I)([\alpha_1, \alpha_2] \ll_{L^I}$

$[0.2, 0.5] \Rightarrow (\exists x \in T_0(-d_B) \cap d_A)([\alpha_1, \alpha_2] \ll_{L^I} \mathcal{C}_{\min}(B(y-x), A(x))))$, from which it follows that $\sup\{[\alpha_1, \alpha_2] \in L^I \mid (\exists x \in T_0(-d_B) \cap d_A)([\alpha_1, \alpha_2] \ll_{L^I} \mathcal{C}_{\min}(B(y-x), A(x)))\} = [0.2, 0.5]$.

Remark that if $A(0)$ would have been smaller or equal to $[0, 0.5]$, then we would have had an equality.

$\diamondsuit$

The construction principle can not be used to extend the binary erosion to an interval-valued fuzzy morphological operator, since it is not increasing w.r.t. the structuring element. Similarly as for weak $[\alpha_1, \alpha_2]$-cuts, it can however be constructed by duality properties.

The interval-valued fuzzy opening and closing can then be constructed as a combination of the interval-valued fuzzy dilation and erosion.

4. Construction of Interval-valued Fuzzy Morphological Operators - Discrete Case

In practice, we do not work in a continuous framework, since any device performs a twofold sampling on the considered images and structuring elements: (i) To make it possible to store an image, its domain is sampled down to a matrix with a given number of rows and columns. In other words, the image domain is sampled down from $\mathbb{R}^n$ to $\mathbb{Z}^n$. (ii) Further, also the grey values can no longer take every possible value in the unit interval $[0, 1]$, but they are limited to a finite subchain of it. As a consequence in interval-valued fuzzy morphology we will no longer work in the continuous chain L^I but in a finite subchain $L^I_{r,s}$, defined by $L^I_{r,s} = \{[\frac{r-k}{r-1}, \frac{s-l}{s-1}] \mid k, l \in \mathbb{Z}$ and $1 \leq k \leq r$ and $1 \leq l \leq s\}$ for given integers r and s. Note that in practice usually

$r = s$, since on the same device the number of bits used to represent the grey value for the lower border and for the upper border of the grey interval will be the same. The image A and the structuring element B now thus belong to $\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, i.e. the class of all interval-valued fuzzy sets in $\mathbb{Z}^n$ with membership intervals in $L^I_{r,s}$. For such sets A and B it holds that $\forall x \in \mathbb{Z}^n$, $A_1(x)$ and $B_1(x)$ belong to $I_r = \{\frac{r-k}{r-1} \mid k \in \mathbb{Z} \text{ and } 1 \leq k \leq r\}$ and analogously $A_2(x)$ and $B_2(x)$ belong to $I_s\{\frac{s-l}{s-1} \mid l \in \mathbb{Z} \text{ and } 1 \leq l \leq s\}$.

The definitions of negators, conjunctors and implicators on the chain $L^I_{r,s}$ can be adopted from the continuous framework by replacing L^I by $L^I_{r,s}$. Remark however that not every operator on L^I has a discrete counterpart. The conjunctor $\mathcal{C}$ given by $\mathcal{C}(x, y) = [x_1 \cdot y_1, x_2 \cdot y_2]$ for all $x, y \in L^I$, for example, is not defined on $L^I_{r,s}$ because the product of two elements of I_r (respectively I_s) does not necessarily belong to I_r (respectively I_s) again.

For the discrete interval-valued fuzzy dilation and erosion we now get the following definitions:

Definition 9. *Let $\mathcal{C}$ be a conjunctor on $L^I_{r,s}$, let $\mathcal{I}$ be an implicator on $L^I_{r,s}$ and let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$. The discrete interval-valued dilation $D^I_{\mathcal{C}}(A, B) \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$ and the discrete interval-valued erosion $E_{\mathcal{I}}(A, B) \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$ are respectively defined as:*

$$\begin{aligned} D^I_{\mathcal{C}}(A, B)(y) &= \sup_{x \in T_y(-d_B) \cap d_A} \mathcal{C}(B(y - x), A(x)) \\ &= [\max_{x \in T_y(-d_B) \cap d_A} \mathcal{C}(B(y - x), A(x))_1, \max_{x \in T_y(-d_B) \cap d_A} \mathcal{C}(B(y - x), A(x))_2]. \end{aligned}$$

$$\begin{aligned} E^I_{\mathcal{I}}(A, B)(y) &= \inf_{x \in T_y(d_B)} \mathcal{I}(B(x - y), A(x)) \\ &= [\min_{x \in T_y(d_B)} \mathcal{I}(B(x - y), A(x))_1, \min_{x \in T_y(d_B)} \mathcal{I}(B(x - y), A(x))_2]. \end{aligned}$$

Remark that for $y \notin D(d_A, d_B)$ it will hold that $D^I_{\mathcal{C}}(A,B)(y) = 0_{L^I}$. Similarly as in the continuous framework, the interval-valued fuzzy opening and closing are formed as a combination of a dilation and an erosion.

We will now investigate the construction of interval-valued fuzzy morphological operators from the corresponding binary operators in the above introduced discrete framework. It will be seen that the characterization of the supremum in the discrete case has as a consequence that some of the difficulties from the continuous case do not arise anymore. Moreover, also some stronger relationships will hold. We will use the notation $U_{L^I_{r,s}}$ for the discrete counterpart of the set U_{L^I}: $U_{L^I_{r,s}} = \{[x_1, x_2] \in L^I_{r,s} \mid x_2 = 1\}$.

4.1. Construction based on weak $[\alpha_1, \alpha_2]$-cuts

Lemma 5. *Let $A \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$. It holds that $A = \bigcup_{[\alpha_1,\alpha_2] \in L^I_{r,s} \setminus \{0_{L^I}\}} [\alpha_1, \alpha_2] A^{\alpha_2}_{\alpha_1}$, i.e., $\forall x \in \mathbb{Z}^n$:*

$$\begin{aligned}
A(x) &= \sup_{[\alpha_1,\alpha_2] \in L^I_{r,s} \setminus \{0_{L^I}\}} ([\alpha_1, \alpha_2] A^{\alpha_2}_{\alpha_1})(x) \\
&= \sup \{[\alpha_1, \alpha_2] \in L^I_{r,s} \setminus \{0_{L^I}\} \mid x \in A^{\alpha_2}_{\alpha_1}\} \\
&= [\max\{\alpha_1 \mid (\exists \alpha_2 \in [\alpha_1, 1] \setminus \{0_{L^I}\})([\alpha_1, \alpha_2] \in L^I_{r,s} \textit{ and } x \in A^{\alpha_2}_{\alpha_1})\}, \\
&\quad \max\{\alpha_2 \mid (\exists \alpha_1 \in [0, \alpha_2])([\alpha_1, \alpha_2] \in L^I_{r,s} \textit{ and } x \in A^{\alpha_2}_{\alpha_1})\}].
\end{aligned}$$

Proof. Similar to the proof of Lemma 1. □

If we now consider a family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2] \in L^I_{r,s} \setminus \{0_{L^I}\}}$ of crisp subsets of $\mathbb{Z}^n$ that is decreasing and we define the interval-valued fuzzy set R in $\mathbb{Z}^n$ for all $x \in \mathbb{Z}^n$ as,

$$\begin{aligned}
R(x) &= \sup_{[\alpha_1,\alpha_2] \in L^I_{r,s} \setminus \{0_{L^I}\}} ([\alpha_1, \alpha_2] P_{[\alpha_1,\alpha_2]})(x) && (16) \\
&= \sup \{[\alpha_1, \alpha_2] \in L^I_{r,s} \setminus \{0_{L^I}\} \mid x \in P_{[\alpha_1,\alpha_2]}\},
\end{aligned}$$

then we might wonder whether it holds that $(\forall[\alpha_1,\alpha_2]\in L^I_{r,s}\backslash\{0_{L^I}\})(R^{\alpha_2}_{\alpha_1}=P_{[\alpha_1,\alpha_2]})$. Just as in the continuous case, the following inclusion always holds:

Proposition 11. *For a decreasing family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I_{r,s}\backslash\{0_{L^I}\}}$ of crisp subsets of $\mathbb{Z}^n$ and the interval-valued fuzzy set R defined in (16), it holds that:*

$$(\forall[\alpha_1,\alpha_2]\in L^I\backslash\{0_{L^I}\})(P_{[\alpha_1,\alpha_2]}\subseteq R^{\alpha_2}_{\alpha_1}).$$

Proof. Analogous to the proof of Proposition 1. □

The following lemma gives us a condition such that the reverse inclusion would also hold.

Lemma 6. *For a decreasing family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I_{r,s}\backslash\{0_{L^I}\}}$ of crisp subsets of $\mathbb{Z}^n$, it holds that*

$$\begin{gathered}(\forall[\alpha_1,\alpha_2]\in L^I_{r,s}\backslash\{0_{L^I}\})(\forall x\in\mathbb{Z}^n)\\ [([\alpha_1,\alpha_2]\in\{[\beta_1,\beta_2]\in L^I_{r,s}\backslash\{0_{L^I}\}\mid x\in P_{[\beta_1,\beta_2]}\}\Leftrightarrow\\ \sup\{[\beta_1,\beta_2]\in L^I_{r,s}\backslash\{0_{L^I}\}\mid x\in P_{[\beta_1,\beta_2]}\}\geq_{L^I}[\alpha_1,\alpha_2])\\ \Updownarrow\\ [\widetilde{SC}]:\Big(\forall[\alpha_1,\alpha_2]\in L^I_{r,s}\backslash\{0_{L^I}\}\Big)\Big(\forall x\in\mathbb{Z}^n\Big)\Big(x\notin P_{[\alpha_1,\alpha_2]}\Rightarrow\\ \big((\forall[\beta_1,\beta_2]\in L^I_{r,s}\backslash\{0_{L^I}\})((\beta_1<\alpha_1\ \textit{and}\ \beta_2\geq\alpha_2)\Rightarrow x\notin P_{[\beta_1,\beta_2]})\big)\ \textit{or}\\ \big((\forall[\beta_1,\beta_2]\in L^I_{r,s}\backslash\{0_{L^I})((\beta_1\geq\alpha_1\ \textit{and}\ \beta_2<\alpha_2)\Rightarrow x\notin P_{[\beta_1,\beta_2]}))\Big).\end{gathered}$$

Proof. Analogous to the proof of Lemma 2. □

The following proposition is a straightforward consequence of the above lemma and Proposition 11.

Proposition 12. *For a decreasing family* $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I_{r,s}\setminus\{0_{L^I}\}}$ *of crisp subsets of* $\mathbb{Z}^n$ *that satisfies* $[\widetilde{SC}]$ *and the interval-valued fuzzy set* R *defined in (16), it holds that:*

$$(\forall[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\})(P_{[\alpha_1,\alpha_2]}=R^{\alpha_2}_{\alpha_1}).$$

Proof. Follows from the proof of Proposition 11 by using Lemma 6. □

Remark that if the decreasing family $(P_{[\alpha_1,\alpha_2]})_{[\alpha_1,\alpha_2]\in L^I_{r,s}\setminus\{0_{L^I}\}}$ does not satisfy the condition $[\widetilde{SC}]$, it will not hold that $(\forall[\alpha_1,\alpha_2]\in L^I\setminus\{0_{L^I}\})(P_{[\alpha_1,\alpha_2]}=R^{\alpha_2}_{\alpha_1})$, with the set R as defined in (16). Indeed, if $[\widetilde{SC}]$ does not hold, then $(\exists[\alpha_1,\alpha_2]\in L^I_{r,s}\setminus\{0_{L^I}\})(\exists x\in\mathbb{Z}^n)(x\notin P_{[\alpha_1,\alpha_2]}$ and $(\exists[\beta_1,\beta_2]\in L^I_{r,s}\setminus\{0_{L^I}\})(\beta_1<\alpha_1$ and $\beta_2\geq\alpha_2$ and $x\in P_{[\beta_1,\beta_2]})$ and $(\exists[\gamma_1,\gamma_2]\in L^I_{r,s}\setminus\{0_{L^I}\})(\gamma_1\geq\alpha_1$ and $\gamma_2<\alpha_2$ and $x\in P_{[\gamma_1,\gamma_2]}))$. This would mean that $R_1(x)\geq\gamma_1\geq\alpha_1$ and $R_2(x)\geq\beta_2\geq\alpha_2$. As a consequence, $x\in R^{\alpha_2}_{\alpha_1}$ and $x\notin P_{[\alpha_1,\alpha_2]}$.

The constructions made in the continuous case can also be done in the discrete case with the same results. Some remarks need however to be given.

Proposition 13. *Let* $A,B\in\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$*, then for all* $y\in\mathbb{Z}^n$ *it holds that:*

$$\widetilde{D(A,B)}(y)=\sup_{x\in T_y(-d_B)\cap d_A}\mathcal{C}_{\min}(B(y-x),A(x))=D^I_{\mathcal{C}_{\min}}(A,B)(y).$$

Proof. The proof is similar to the one of Proposition 5, where it has to be shown that $(*)=(**)$, with $(*)$ and $(**)$ given by:

$$\begin{aligned}(*)=\sup\{&[\alpha_1,\alpha_2]\in L^I_{r,s}\setminus\{0_{L^I}\}\mid\\&(\exists x\in T_y(-d_B)\cap d_A)(\mathcal{C}_{\min}(B(y-x),A(x))\geq_{L^I}[\alpha_1,\alpha_2])\},\end{aligned}$$

$$(**)=\sup_{x\in T_y(-d_B)\cap d_A}\mathcal{C}_{\min}(B(y-x),A(x))).$$

The proof of $(*) \leq (**)$ is analogous to the proof of Proposition 5. The proof of $(**) \leq (*)$ however is much simpler in the discrete framework, since we do not have to make use of the characterization of the supremum. If $T_y(-d_B) \cap d_A = \emptyset$, then $(**) = 0_{L^I}$ and thus $(**) \leq_{L^I} (*)$. Otherwise, in the discrete case, it immediately follows from $(**) = \sup\limits_{x \in T_y(-d_B) \cap d_A} \mathcal{C}_{\min}(B(y-x), A(x)))$ that

$$\begin{aligned}
& (**)_1 \in \{\alpha_1 \mid (\exists \alpha_2 \in [\alpha_1, 1])([\alpha_1, \alpha_2] \in L^I_{r,s} \backslash \{0_{L^I}\} \text{ and} \\
& (\exists x \in T_y(-d_B) \cap d_A)([\alpha_1, \alpha_2] \leq_{L^I} \mathcal{C}_{\min}(B(y-x), A(x))))\} \text{ and} \\
& (**)_2 \in \{\alpha_2 \mid (\exists \alpha_1 \in [0, \alpha_2])([\alpha_1, \alpha_2] \in L^I_{r,s} \backslash \{0_{L^I}\} \text{ and} \\
& (\exists x \in T_y(-d_B) \cap d_A)([\alpha_1, \alpha_2] \leq_{L^I} \mathcal{C}_{\min}(B(y-x), A(x))))\} \\
\Rightarrow \quad & [(**)_1, (**)_2] \leq_{L^I} (*)
\end{aligned}$$

$\square$

Similarly to the continuous case, an interval-valued fuzzy erosion, opening and closing can then be constructed by using duality properties.

4.2. Construction based on strict $[\alpha_1, \alpha_2]$-cuts

Because of the similarity to the case of weak $[\alpha_1, \alpha_2]$-cuts, we will leave the proofs in this subsection to the reader.

We first determine the unit e_r (respectively e_s) of the finite chain $I_r = \{0, \frac{1}{r-1}, \ldots, \frac{r-2}{r-1}, 1\}$ (respectively $I_s = \{0, \frac{1}{s-1}, \ldots, \frac{s-2}{s-1}, 1\}$) as $e_r = \frac{1}{r-1}$ (respectively $e_s = \frac{1}{s-1}$). We assume that $e_r = e_s$, which is usually the case in practice. Further, the sum of (respectively difference between) an intervals $[x_1, x_2]$ and $[e_r, e_s]$ is given by $[x_1 + e_r, x_2 + e_s]$ (respectively $[x_1 - e_r, x_2 -$

$e_s]$). The assumption $e_r = e_s$ is needed if we want $[x_1 + e_r, x_2 + e_s]$ and $[x_1 - e_r, x_2 - e_s]$ to be intervals. Additionally, we define the set $G_{r,s}$ by $G_{r,s} = \{[\alpha_1, \alpha_2] \mid (\alpha_1 = -e_r \text{ and } \alpha_2 \in (I_s \setminus \{1\}) \cup \{-e_s\}))\}$. Remark that $G_{r,s} \cap L^I_{r,s} = \emptyset$. Finally, we extend the order relation on $L^I_{r,s}$ to $L^I_{r,s} \cup G_{r,s}$ in a straightforward manner and for this reason, we will use the same notation $\leq_{L^I}$:

$$x \leq_{L^I} y \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2,\ \forall x, y \in L^I_{r,s} \cup G_{r,s}. \tag{17}$$

Also the order relation $\ll_{L^I}$ is extended analogously. The infimum and supremum of an arbitrary subset S of $L^I_{r,s} \cup G_{r,s}$ are then respectively given by:

$$\inf S = [\inf_{x \in S} x_1, \inf_{x \in S} x_2] = [\min_{x \in S} x_1, \min_{x \in S} x_2], \tag{18}$$

$$\sup S = [\sup_{x \in S} x_1, \sup_{x \in S} x_2] = [\max_{x \in S} x_1, \max_{x \in S} x_2]. \tag{19}$$

We can now formulate the following lemma that resembles Lemma 3, but does differ from it.

Lemma 7. *Let $A \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then it holds $\forall x \in \mathbb{Z}^n$ that:*

$$A(x) = [\max\{\alpha_1 \mid (\exists \alpha_2 \in [\alpha_1, 1[)([\alpha_1, \alpha_2] \in (L^I_{r,s} \backslash U_{L^I_{r,s}}) \cup G_{r,s} \text{ and }$$
$$A_1(x) > \alpha_1 \text{ and } A_2(x) > \alpha_2)\}, \max\{\alpha_2 \mid (\exists \alpha_1 \in [0, \alpha_2])$$
$$([\alpha_1, \alpha_2] \in (L^I_{r,s} \backslash U_{L^I_{r,s}}) \cup G_{r,s} \text{ and } A_1(x) > \alpha_1 \text{ and } A_2(x) > \alpha_2)\}] + [e_r, e_s].$$

As a consequence, we have to take into account the interval $[e_r, e_s]$ also for the construction of interval-valued fuzzy operators. If we now consider a family $(Q_{\overline{[\alpha_1, \alpha_2]}})_{[\alpha_1,\alpha_2] \in (L^I_{r,s} \backslash U_{L^I_{r,s}}) \cup G_{r,s}}$ of crisp subsets of $\mathbb{Z}^n$ that is decreasing and we define the interval-valued fuzzy set V in $\mathbb{Z}^n$ as,

$$V(x) = \sup\{[\alpha_1, \alpha_2] \in (L^I_{r,s} \setminus U_{L^I_{r,s}}) \cup G_{r,s} \mid x \in Q_{\overline{[\alpha_1, \alpha_2]}}\} + [e_r, e_s], \forall x \in \mathbb{Z}^n, \tag{20}$$

then we might wonder whether it holds that $(\forall[\alpha_1,\alpha_2] \in L^I_{r,s}\backslash U_{L^I})(V^{\overline{\alpha_2}}_{\overline{\alpha_1}} = Q_{\overline{[\alpha_1,\alpha_2]}})$. Remark that nonetheless the fact that V is for all $x \in \mathbb{Z}^n$ constructed as the supremum of a set in $(L^I_{r,s} \setminus U_{L^I_{r,s}}) \cup G_{r,s}$, $V(x)$ will always belong $L^I_{r,s}$.

In contrast to the continuous case, the inclusion $Q_{\overline{[\beta_1,\beta_2]}} \subseteq V^{\overline{\beta_2}}_{\overline{\beta_1}}$ always holds.

Proposition 14. *For a decreasing family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2]\in(L^I_{r,s}\backslash U_{L^I_{r,s}})\cup G_{r,s}}$ of crisp subsets of $\mathbb{Z}^n$ and the interval-valued fuzzy set V defined in (20), it holds that:*

$$(\forall[\alpha_1,\alpha_2] \in L^I_{r,s}\backslash U_{L^I_{r,s}})(Q_{\overline{[\alpha_1,\alpha_2]}} \subseteq V^{\overline{\alpha_2}}_{\overline{\alpha_1}}).$$

The following lemma gives us a condition such that the reverse inclusion would also hold.

Lemma 8. *For a decreasing family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2]\in(L^I_{r,s}\backslash U_{L^I_{r,s}})\cup G_{r,s}}$ of crisp subsets of $\mathbb{Z}^n$, it holds that*

$$(\forall[\alpha_1,\alpha_2] \in (L^I_{r,s} \setminus U_{L^I_{r,s}}) \cup G_{r,s})(\forall x \in \mathbb{Z}^n)$$
$$([\alpha_1,\alpha_2] \in \{[\beta_1,\beta_2] \in (L^I_{r,s} \setminus U_{L^I_{r,s}}) \cup G_{r,s} \mid x \in Q_{\overline{[\beta_1,\beta_2]}}\} \Leftrightarrow$$
$$\sup\{[\beta_1,\beta_2] \in (L^I_{r,s} \setminus U_{L^I_{r,s}}) \cup G_{r,s} \mid x \in Q_{\overline{[\beta_1,\beta_2]}}\} \geq_{L^I} [\alpha_1,\alpha_2])$$
$$\Updownarrow$$
$$[\widetilde{SC'}] : \Big(\forall[\alpha_1,\alpha_2] \in (L^I_{r,s} \setminus U_{L^I_{r,s}}) \cup G_{r,s}\Big)\Big(\forall x \in \mathbb{Z}^n\Big)\Big(x \notin Q_{\overline{[\alpha_1,\alpha_2]}} \Rightarrow$$
$$\big((\forall[\beta_1,\beta_2] \in (L^I_{r,s} \setminus U_{L^I_{r,s}}) \cup G_{r,s})((\beta_1 < \alpha_1 \textit{ and } \beta_2 \geq \alpha_2) \Rightarrow x \notin Q_{\overline{[\beta_1,\beta_2]}})\big) \textit{ or}$$
$$\big((\forall[\beta_1,\beta_2] \in (L^I_{r,s} \setminus U_{L^I_{r,s}}) \cup G_{r,s})((\beta_1 \geq \alpha_1 \textit{ and } \beta_2 < \alpha_2) \Rightarrow x \notin Q_{\overline{[\beta_1,\beta_2]}})\big)\Big).$$

The following proposition is a straightforward consequence of the above lemma and Proposition 14.

Proposition 15. *For a decreasing family* $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2]\in(L^I_{r,s}\setminus U_{L^I_{r,s}})\cup G_{r,s}}$ *of crisp subsets of* $\mathbb{Z}^n$ *that satisfies* $[\widetilde{SC'}]$ *and the interval-valued fuzzy set* V *defined in (20), it holds that:*

$$(\forall[\alpha_1,\alpha_2]\in L^I_{r,s}\setminus U_{L^I_{r,s}})(Q_{\overline{[\alpha_1,\alpha_2]}}=V^{\overline{\alpha_2}}_{\overline{\alpha_1}}).$$

Remark that if a decreasing family $(Q_{\overline{[\alpha_1,\alpha_2]}})_{[\alpha_1,\alpha_2]\in(L^I_{r,s}\setminus U_{L^I_{r,s}})\cup G_{r,s}}$ of crisp subsets of $\mathbb{Z}^n$ does not satisfy condition $[\widetilde{SC'}]$, then it will also not hold that $(\forall[\alpha_1,\alpha_2]\in L^I_{r,s}\setminus U_{L^I_{r,s}})(Q_{\overline{[\alpha_1,\alpha_2]}}=V^{\overline{\alpha_2}}_{\overline{\alpha_1}})$. Indeed, if $[\widetilde{SC'}]$ does not hold, then $(\exists[\alpha_1,\alpha_2]\in(L^I_{r,s}\setminus U_{L^I_{r,s}})\cup G_{r,s})(\exists x\in\mathbb{Z}^n)(x\notin Q_{\overline{[\alpha_1,\alpha_2]}}$ and $(\exists[\beta_1,\beta_2]\in(L^I_{r,s}\setminus U_{L^I_{r,s}})\cup G_{r,s})(\beta_1<\alpha_1$ and $\beta_2\geq\alpha_2$ and $x\in Q_{\overline{[\beta_1,\beta_2]}})$ and $(\exists[\gamma_1,\gamma_2]\in(L^I_{r,s}\setminus U_{L^I_{r,s}})\cup G_{r,s})(\gamma_1\geq\alpha_1$ and $\gamma_2<\alpha_2$ and $x\in Q_{\overline{[\gamma_1,\gamma_2]}}))$. This would mean that $V_1(x)\geq\gamma_1+e_r>\alpha_1$ and $V_2(x)\geq\beta_2+e_s>\alpha_2$. Since $\gamma_2<\alpha_2$ and $\beta_1<\alpha_1$, $[\alpha_1,\alpha_2]\notin G_{r,s}$, but $[\alpha_1,\alpha_2]\in L^I_{r,s}$. As a consequence, $x\in V^{\overline{\alpha_2}}_{\overline{\alpha_1}}$ and $x\notin Q_{\overline{[\alpha_1,\alpha_2]}}$.

For the construction of the interval-valued fuzzy dilation by strict $[\alpha_1,\alpha_2]$-cuts, we find a stronger result in the discrete case than in the continuous case. We first need to extend the definition of strict $[\alpha_1,\alpha_2]$-cuts from $(L^I_{r,s}\setminus U_{L^I_{r,s}})$ to $(L^I_{r,s}\setminus U_{L^I_{r,s}})\cup G_{r,s}$ as follows. For $A\in\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$ and $[\alpha_1,\alpha_2]\in G_{r,s}$,

$$A^{\overline{\alpha_2}}_{\overline{\alpha_1}}=\begin{cases}\mathbb{Z}^n & \alpha_1=-e_r \text{ and } \alpha_2=-e_s\\ A^{\overline{\alpha_2}} & \alpha_1=-e_r \text{ and } \alpha_2\neq-e_s\end{cases}.$$

We define $\widetilde{D(A,B)}'$ for all $x\in\mathbb{Z}^n$ as

$$\widetilde{D(A,B)}'(x)=\sup\{[\alpha_1,\alpha_2]\in(L^I_{r,s}\setminus U_{L^I_{r,s}})\cup G_{r,s}\mid x\in D(A^{\overline{\alpha_2}}_{\overline{\alpha_1}},B^{\overline{\alpha_2}}_{\overline{\alpha_1}})\}+[e_r,e_s].$$

Remark that $\widetilde{D(A,B)}'(x)\in L^I_{r,s}$ for all $x\in\mathbb{Z}^n$.

The following proposition states that the above constructed dilation $\widetilde{D(A,B)}'$ equals the dilation $D^I_{\mathcal{C}_{\min}}$.

Proposition 16. *Let $A, B \in \mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$, then for all $y \in \mathbb{Z}^n$ it holds that:*

$$\widetilde{D(A,B)}'(y) = \sup_{x \in T_y(-d_B) \cap d_A} \mathcal{C}_{\min}(B(y-x), A(x)) = D^I_{\mathcal{C}_{\min}}(A,B)(y).$$

Similarly to the continuous case, an interval-valued fuzzy erosion, opening and closing can then be constructed by using duality properties.

5. Conclusion

In this paper we have studied the construction of increasing interval-valued fuzzy operators from their corresponding binary counterparts, in particular the construction of morphological operators that are increasing w.r.t. to both the image and structuring element. This construction was investigated both in the general continuous case and the practical discrete case. In this discrete case, we work with interval-valued fuzzy sets from $\mathcal{IVFS}_{r,s}(\mathbb{Z}^n)$ instead of $\mathcal{IVFS}(\mathbb{R}^n)$ since in practice, both the image domain and the range of grey values are sampled due to technical limitations. It was shown that in both cases the constructed interval-valued fuzzy dilation corresponds to the interval-valued fuzzy dilation $D^I_{\mathcal{C}_{min}}$, that is dual to the erosion $E_{\mathcal{I}_{\min},\mathcal{N}_s}$, which allows us to construct the other basic morphological operators. Further, we found out that the characterization of the supremum in the discrete case circumvents some of the difficulties from the continuous case. Moreover, some stronger relationships hold in this practical case. A drawback of the discrete framework is that not all operators from the continuous framework have a discrete counterpart.

References

[1] H. Heijmans, Morphological image operators, Advances in Electronics and Electron Physics Series, Academic Press, Boston, 1994.

[2] L. Najman and H. Talbot (eds.), Mathematical Morphology: from theory to applications, ISTE-Wiley, London, UK, ISBN 9781848212152, Sept. 2010.

[3] J. Serra, Image Analysis and Mathematical Morphology - Volume II : Theoretical Advances, Academic Press, London, 1988.

[4] J. Serra, Image Analysis and Mathematical Morphology, Academic Press, London, 1982.

[5] R.M. Haralick, R.S. Sternberg and X. Zhuang, Image analysis using mathematical morphology, IEEE Trans. Pattern Anal. Mach. Intell. 9(4) (1987) 532-550.

[6] B. De Baets, Fuzzy Morphology: a logical approach, in: B.M. Ayyub and M.M. Gupta (Eds.), Uncertainty Analysis in Engineering and Sciences: Fuzzy Logic, Statistics, and Neural Network Approach, Kluwer Academic Publishers, Boston, 1997, pp. 53-67.

[7] J. Angulo, Morphological colour operators in totally ordered lattices based on distances. Application to image filtering, enhancement and analysis, Computer Vision and Image Understanding, 107(2-3) (2007) 56-73.

[8] J. Chanusot and P. Lambert, Total ordering based on space filling curves for multivalued morphology, in: Proc. of the 4th Intl. Symposium on Mathematical Morphology and its applications (ISSM98), pages 5158, Amsterdam, The Netherlands, June 1998.

[9] I. Bloch, Mathematical Morphology on Bipolar Fuzzy Sets, in: Proc. of International Symposium on Mathematical Morphology, ISMM, 2007, pp. 3-4.

[10] I. Bloch, Dilation and erosion of spatial bipolar fuzzy sets, in: Proc. WILF, 2007, LNAI, vol. 4578, pp. 385-393.

[11] M. Nachtegael, P. Sussner, T. Mélange and E.E. Kerre, Some aspects of interval-valued and intuitionistic fuzzy mathematical morphology, in: Proc. of 3rd International Conference on Image Processing, Computer Vision and Pattern Recognition, IPCV, 2008, pp. 538-543.

[12] M. Nachtegael, P. Sussner, T. Mélange and E.E. Kerre, On the role of complete lattices in mathematical morphology: From tool to uncertainty model, Information Sciences, accepted.

[13] M. Nachtegael, P. Sussner, T. Mélange and E.E. Kerre, Modelling numerical and spatial uncertainty in grayscale image capture using fuzzy set theory, in: Proc. of 1st International North American Simulation Technology Conference, NASTEC, 2008, pp. 15-22.

[14] K. Atanassov, Intuitionistic fuzzy sets, Heidelberg, Germany: Physica Verlag, 1999.

[15] G. Deschrijver and E.E. Kerre, On the relationship between some extensions of fuzzy set theory, Fuzzy Sets Syst. 133 (2003) 227-235.

[16] S.D. Cabrera, K. Iyer, G. Xiang and V. Kreinovich, On inverse halftoning: computational complexity and interval computations, in: Proc. CISS, 2005, paper 164.

[17] A.E. Brito and O. Kosheleva, Interval+image=wavelet: for image processing under interval uncertainty, wavelets are optimal, Reliable Computing 4(4) (1998) 771-783.

[18] E. Barrenechea, Image processing with interval-valued fuzzy sets - edge detection - contrast, Ph.D.thesis, Public university of Navarra, 2005.

[19] T. Mélange, M. Nachtegael, P. Sussner and E.E. Kerre, On the decomposition of interval-valued fuzzy morphological operators, Journal of Mathematical Imaging and Vision, 36(4) (2010) 270-290 DOI:10.1007/s10851-009-0185-7.

[20] R. Sambuc, Fonctions Φ-floues. Application à l'aide au diagnostic en pathologie thyroidienne, Ph.D. thesis, Univ. Marseille, France, 1975.

[21] L.A. Zadeh, Fuzzy Sets, Information and Control 8 (1965) 338-353.

[22] G. Deschrijver and C. Cornelis, Representability in Interval-valued Fuzzy Set Theory, Int. J. Uncertainty Fuzziness and Knowl.-Based Syst. 15(3) (2007) 345-361.

[23] G. Wang and X. Li, The applications of interval-valued fuzzy numbers and interval-distribution numbers, Fuzzy Sets Syst. 98 (1998) 331-335.

[24] T. Mélange, M. Nachtegael, P. Sussner and E.E. Kerre, Basic Properties of the Interval-valued Fuzzy Morphological Operators, in: Proc. FUZZ-IEEE 2010, pp. 822-829.